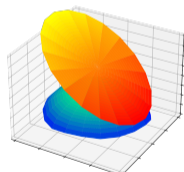
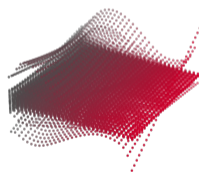
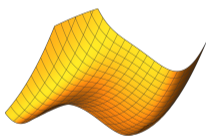
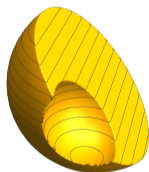


# Accurately and efficiently solving structured nonconvex optimization problems

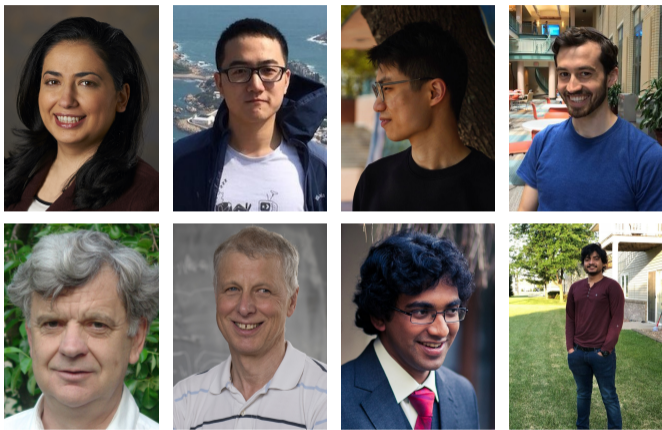
**Alex L. Wang**

Carnegie Mellon University



These slides are publicly available at [cs.cmu.edu/~alw1](http://cs.cmu.edu/~alw1)

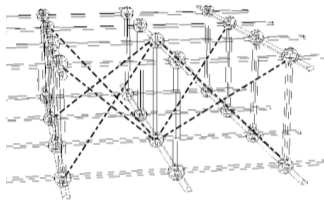
# Collaborators



Carnegie Mellon University (OR, Math, CS),  
Northwestern University, Fudan University,  
Peking University

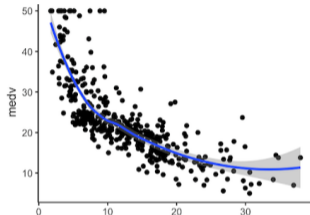
# Convex optimization

- Convex optimization is influential in many different fields



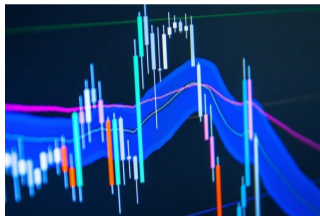
## Engineering

Controller stability, power allocation, truss design, +



## Statistics

(Linear) Regression, parameter estimation, +



## Finance

Portfolio optimization, risk analysis, +

- Convex optimization is accurate and efficient

## Convex optimization, meet nonconvex problems

- Unfortunately, many practical optimization problems are **nonconvex**
- Example: Low-rank matrix completion (**Netflix problem**)

	Movies				
Users	5	4	?	?	4
	3	?	?	3	?
	?	2	4	1	1
	?	3	?	?	4

$$\min_{X \in \mathbb{R}^{n \times k}} \{\text{rank}(X) : X \text{ agrees with revealed entries}\}$$

- Rank constraints, binary constraints, sparsity constraints
- Generally hard, **but not always!**
- Some nonconvex problems can be solved using convex optimization

### Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

- **Completed work:**

- Nonconvex problems: quadratically constrained quadratic programs (QCQPs)
- Convex relaxations: semidefinite programs (SDPs)

### Today's questions

Understand **structures within QCQPs** that enable us to solve them **exactly and efficiently** using **SDPs**

- **Preliminaries**

QCQPs and their applications, the SDP relaxation

- **Understand structures within QCQPs that enable us to solve them. . .**

- **exactly** [\[IPCO 20\]](#), [\[Math. Prog. 21\]](#), [\[Math. Prog. \*under review\*\]](#)

Objective value, convex hull exactness, applications

- **efficiently** [\[Math. Prog. 20\]](#), [\[SIAM J. Optim. \*under review\*\]](#), [\[Ongoing\]](#)

The generalized trust-region subproblem and regular QCQPs

- **Conclusion and future directions**

# 1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

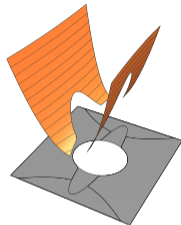
4 Conclusion and future directions

# Quadratically constrained quadratic programs (QCQPs)

- $q_{\text{obj}}, q_1, \dots, q_m : \mathbb{R}^n \rightarrow \mathbb{R}$  quadratic (possibly nonconvex!)

$$q_i(x) = x^\top A_i x + 2b_i^\top x + c_i$$

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_i(x) \leq 0, \forall i \in [m]\}$$



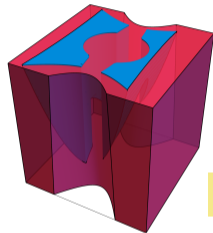
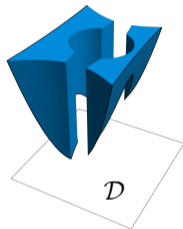
- Highly expressive:
  - MAX-CUT, MAX-CLIQUE, pooling, truss design, facility location, production planning
  - binary program  $x_1(1 - x_1) = 0$
  - polynomial optimization problems  $x_1 x_2 = z_{12}$
- NP-hard in general



# The QCQP epigraph

- QCQP epigraph

$$\mathcal{D} := \left\{ (x, t) \in \mathbb{R}^{n+1} : \begin{array}{l} q_{\text{obj}}(x) \leq t \\ q_i(x) \leq 0, \forall i \in [m] \end{array} \right\}$$



$$q(\gamma', x) \leq t$$

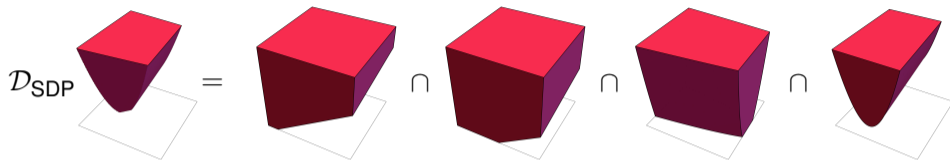
- How can we derive ~~convex~~ relaxations of  $\mathcal{D}$ ?

- If  $\gamma \in \mathbb{R}_+^m$ , then

$$\forall (x, t) \in \mathcal{D}, \quad \underbrace{q_{\text{obj}}(x) + \sum_{i=1}^m \gamma_i q_i(x)}_{=: q(\gamma, x)} \leq t$$

# The SDP relaxation

- SDP relaxation = impose all convex aggregated inequalities!



- Formally,

$$\Gamma := \{\gamma \in \mathbb{R}_+^m : q(\gamma, x) \text{ is convex in } x\} = \left\{ \gamma \in \mathbb{R}_+^m : A_{\text{obj}} + \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}$$

$$\mathcal{D}_{\text{SDP}} := \bigcap_{\gamma \in \Gamma} \{(x, t) : q(\gamma, x) \leq t\} = \left\{ (x, t) \in \mathbb{R}^{n+1} : \sup_{\gamma \in \Gamma} q(\gamma, x) \leq t \right\}$$

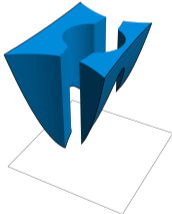
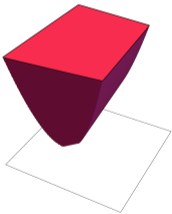
$$\text{Opt}_{\text{SDP}} := \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x)$$

# The usual SDP relaxation

$$\begin{aligned} & \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_i(x) \leq 0, \forall i \in [m]\} \\ &= \inf_{x \in \mathbb{R}^n, X \in \mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X \rangle + 2b_{\text{obj}}^\top x + c_{\text{obj}} : \begin{array}{l} X = xx^\top \\ \langle A_i, X \rangle + 2b_i^\top x + c_i \leq 0, \forall i \in [m] \end{array} \right\} \\ &\geq \inf_{x \in \mathbb{R}^n, X \in \mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X \rangle + 2b_{\text{obj}}^\top x + c_{\text{obj}} : \begin{array}{l} X - xx^\top \succeq 0 \\ \langle A_i, X \rangle + 2b_i^\top x + c_i \leq 0, \forall i \in [m] \end{array} \right\} \\ &= \inf_{x \in \mathbb{R}^n} \inf_{X \in \mathbb{S}^n} \dots \\ &= \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x) \end{aligned}$$

## Preliminaries recap

- Main objects of interest

	nonconvex QCQP	convex SDP
Optimum value	$\text{Opt}$	$\text{Opt}_{\text{SDP}}$
Epigraph	$\mathcal{D}$	$\mathcal{D}_{\text{SDP}}$
		

- Useful for analysis:
  - $q(\gamma, x) =$  Lagrangian function
  - $\Gamma =$  aggregation weights giving convex  $q(\gamma, x)$

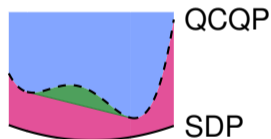
- 1 Preliminaries
- 2 Objective value exactness, convex hull exactness, applications
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- 4 Conclusion and future directions

# Forms of exactness

- What does exactness mean?

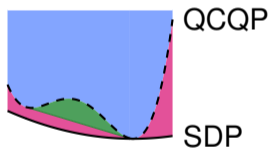
- Objective value exactness:  $\text{Opt} = \text{Opt}_{\text{SDP}}$

- Convex hull exactness:  $\text{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$  ← convexification of substructures



Obj. val. ex.  $\times$

Conv. hull ex.  $\times$



Obj. val. ex.  $\checkmark$

Conv. hull ex.  $\times$

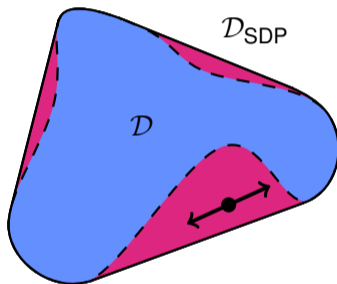


Obj. val. ex.  $\checkmark$

Conv. hull ex.  $\checkmark$

## Convex hull exactness

- $\text{conv}(\mathcal{D}) \stackrel{?}{=} \mathcal{D}_{\text{SDP}}$



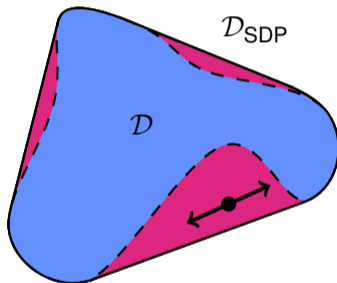
$$\text{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$$



“Given any point in  $\mathcal{D}_{\text{SDP}} \setminus \mathcal{D}$ , exists direction such that can move forward and backward inside  $\mathcal{D}_{\text{SDP}}$ ”

# Sufficient conditions for exactness

- Can carry out this idea for QCQPs!
- Leads to sufficient conditions based on abstract properties
- $\rightarrow$  more concrete conditions



## Theorem ([Math. Prog. 21])

Suppose  $\Gamma$  polyhedral. If for every semidefinite face  $\mathcal{F} \trianglelefteq \Gamma$ ,

$$\text{aff} \left( \text{Proj}_{\mathcal{V}(\mathcal{F})} \{b(\gamma) : \gamma \in \mathcal{F}\} \right) \neq \mathcal{V}(\mathcal{F}),$$

then  $\text{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$ .

## Theorem ([Math. Prog. under review])

If for every  $(x, t) \in \mathcal{D}_{\text{SDP}} \setminus \mathcal{D}$ ,

$$\left\{ (x', t') \in \mathbb{R}^{n+1} : \begin{array}{l} x' \in \ker(A(f)) \\ \langle A(\eta)x + b(\eta), x' \rangle - t' = 0, \forall (1, \eta) \in \mathcal{G}^\perp \end{array} \right\} \neq \{0\},$$

then  $\text{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$ .

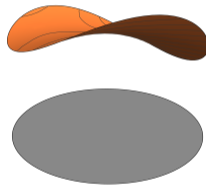
Based on: [\[IPCO 19\]](#), [\[Math. Prog. 21\]](#), [\[Math. Prog. under review\]](#)



## Example: the trust-region subproblem

- Convex hull exactness in the case of **single ball constraint**

$$\text{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ q_{\text{obj}}(x) : \|x\|^2 \leq 1 \right\}$$



- **Applications:**
  - Nonlinear minimization (trust-region methods), combinatorial optimization, robust optimization

---

Based on: [\[IPCO 19\]](#), [\[Math. Prog. 20\]](#)

Related: Yakubovich [1971], Yıldırım [2009], Ho-Nguyen and Kılınç-Karzan [2017]

## Example: QCQPs with symmetry

- Convex hull exactness in the case of “highly symmetric” QCQPs
- Suppose  $A_{\text{obj}} = I_k \otimes \mathbb{A}_{\text{obj}}$ ,  $A_i = I_k \otimes \mathbb{A}_i$  for all  $i \in [m]$

$$I_k \otimes \mathbb{A} = \begin{pmatrix} \mathbb{A} & & & \\ & \mathbb{A} & & \\ & & \ddots & \\ & & & \mathbb{A} \end{pmatrix}$$

and  $k \geq m$

- **Applications:**
  - Robust least squares, sphere packing, QCQPs with spherical constraints, orthogonal Procrustes problem

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Based on: [\[Math. Prog. under review\]](#)

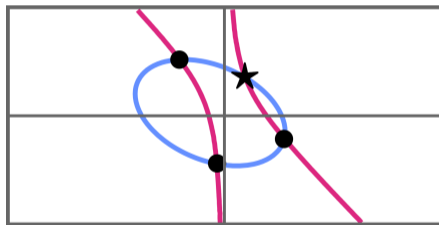
Related: Beck [2007], Beck et al. [2012]

## Example: Random underconstrained quadratic systems

- Obj. val. exactness in the case of random underconstrained quadratic systems

- Solve

$$\inf_{x \in \mathbb{R}^n} \left\{ \|x\|^2 : q_i(x) = 0, \forall i \in [m] \right\}$$



- Fix  $m$ , let  $n \rightarrow \infty$ , if data generated “as Gaussians”, then objective value exactness w.p.  $1 - o(1)$

Based on: [\[Math. Prog. under review\]](#)

Related: Burer and Ye [2019], Locatelli [2020]

## Summary of Part 1

- Sufficient conditions for convex hull exactness
- **Necessary and sufficient** if  $\Gamma$  is polyhedral (dual facially exposed)
- Sufficient conditions for objective value exactness
- Rank-one-generated (ROG) cones: strengthening of convex hull exactness
- **Applications:**
  - Diagonal QCQPs with sign-definite linear terms, semi-random QCQPs, ratios of quadratic functions

### Exactness

[IPCO 20], [Math. Prog. 21],  
[Math. Prog. *under review*]



### ROG Cones

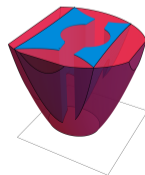
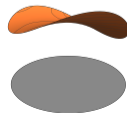
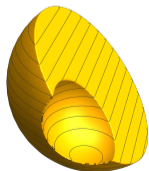
[Tut. Oper. Res. 21],  
[Math. Oper. Res. 21]



### Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

**SDPs provide exact reformulations for broad classes of QCQPs!**



- 1 Preliminaries
- 2 Objective value exactness, convex hull exactness, applications
- 3 Efficient algorithms for structured QCQPs**
- 4 Conclusion and future directions

## Revisiting the SDP relaxation

- Usual SDP relaxation in  $x \in \mathbb{R}^n$  and  $X \in \mathbb{S}^n \implies \approx n^2$  variables
  - Interior point method  $\implies \Theta(n^2 + m^2)$  storage

- Our view: 
$$\text{Opt}_{\text{SDP}} := \inf_{x \in \mathbb{R}^n} \left( \sup_{\gamma \in \Gamma} q(\gamma, x) \right)$$

is a minimization problem in the original space  $\implies n$  variables

# The generalized trust-region subproblem (GTRS)

- Special setting with **single constraint** ( $\leq$  or  $=$ )

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \leq 0\}$$

- **TRS Applications:**

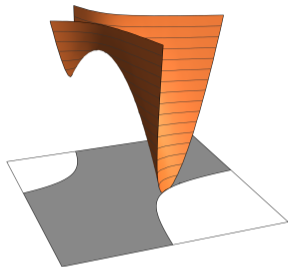
- nonlinear programming (trust-region methods), combinatorial optimization, robust optimization

- **GTRS Applications:**

- minimizing **quartics** of the form  $q(x, p(x))$

$$\inf_{x \in \mathbb{R}^n, \alpha} \{q(x, \alpha) : \alpha = p(x)\}$$

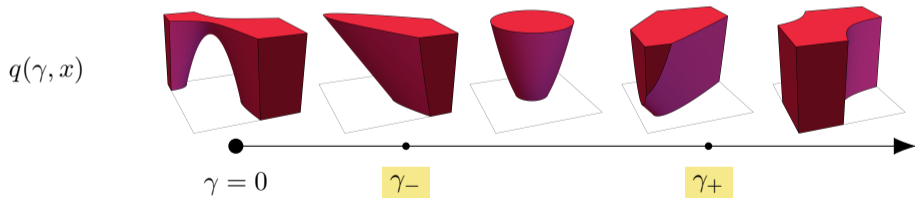
(source localization, constrained rank-one approximation),  
**iterative QCQP solvers**





# Linear-time algorithm for the GTRS

- Convex hull exactness holds  $\text{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}} = \left\{ (x, t) : \sup_{\gamma \in \Gamma} q(\gamma, x) \leq t \right\}$
- Recall  $\Gamma = \{ \gamma \in \mathbb{R}_+ : q(\gamma, x) \text{ is convex in } x \}$

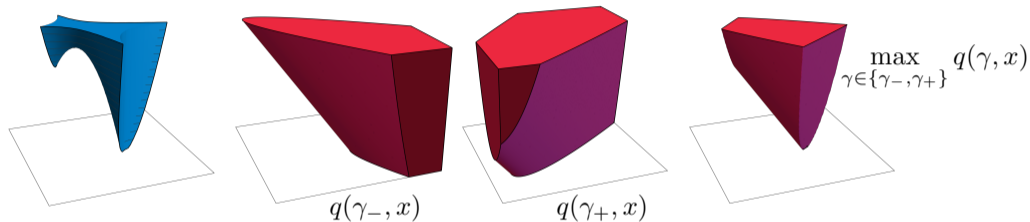


Based on: [\[Math. Prog. 20\]](#)

Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

# Linear-time algorithm for the GTRS

- $\Gamma = [\gamma_-, \gamma_+] \implies \text{Opt} = \text{Opt}_{\text{SDP}} = \inf_{x \in \mathbb{R}^n} \max_{\gamma \in \{\gamma_-, \gamma_+\}} q(\gamma, x)$



- Algorithmic idea**

- Compute  $\gamma_-$  and  $\gamma_+$  to some accuracy
- Apply accelerated gradient descent  $\implies \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \frac{1}{\sqrt{\epsilon}}$

Based on: [\[Math. Prog. 20\]](#)

Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

## Is this running time optimal?

- GTRS:

$$\inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \leq 0\} \quad \tilde{O} \left( \frac{N}{\sqrt{\epsilon}} \log \left( \frac{n}{p} \right) \log \left( \frac{1}{\epsilon} \right) \right)$$

- Minimum eigenvalue:

$$\inf_{x \in \mathbb{R}^n} \{x^T A x : \|x\|^2 = 1\} \quad O \left( \frac{N}{\sqrt{\epsilon}} \log \left( \frac{n}{p} \right) \right)$$

- Smooth convex quadratic:

$$\inf_{x \in \mathbb{R}^n} q_{\text{obj}}(x) \quad O \left( \frac{N}{\sqrt{\epsilon}} \right)$$

- But, these are **hardest problems within the GTRS!**
- Can do better when regularity  $\mu^* > 0$

---

Based on: [\[Math. Prog. 20\]](#)

Related: Kuczynski and Wozniakowski [1992], Nesterov [2018]

# Linear convergence for regular GTRS

- $\mu^* > 0$  holds for most GTRS
- Dual problem

$$\text{Opt}_{\text{SDP}} := \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x) = \sup_{\gamma \in \Gamma} \inf_{x \in \mathbb{R}^n} q(\gamma, x)$$

## Definition

Let  $\gamma^*$  be dual optimizer. Define  $\mu^* := \lambda_{\min}(A_{\text{obj}} + \gamma^* A_1)$ . GTRS instance is regular if  $\mu^* > 0$ .

- Minimum eigenvalue problem is not regular
- Some instances of smooth quadratic minimization are not regular

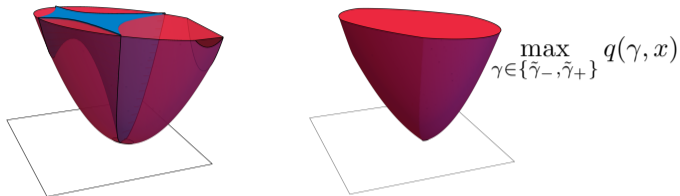
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Based on: [\[SIAM J. Optim. under review\]](#)

Related: Carmon and Duchi [2018]

# Linear convergence for regular GTRS

- Suppose  $\mu^* > 0$



- Suppose  $\gamma^* \in [\tilde{\gamma}_-, \tilde{\gamma}_+] \subseteq \Gamma \implies \text{Opt} = \inf_{x \in \mathbb{R}^n} \max_{\gamma \in \{\tilde{\gamma}_-, \tilde{\gamma}_+\}} q(\gamma, x)$
- Suffices to estimate  $\gamma^*$  roughly and can exploit strong convexity

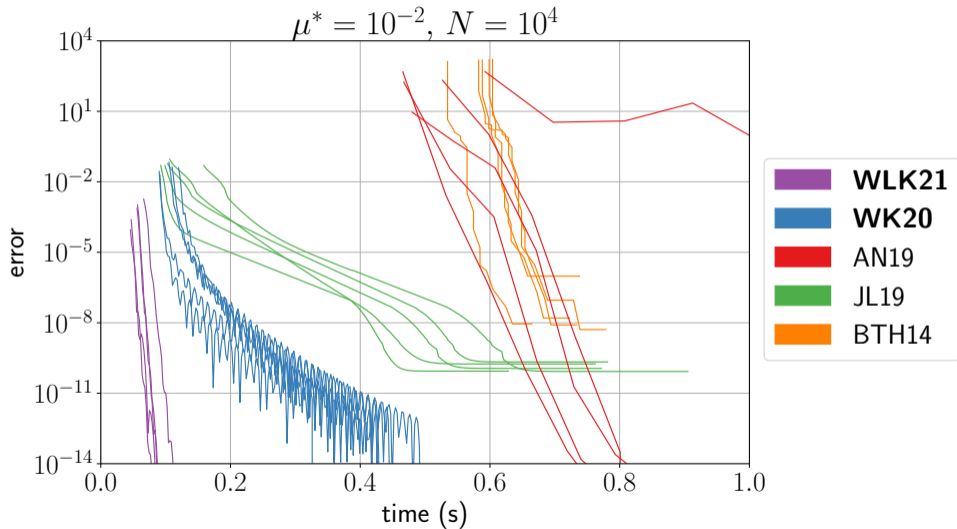
$$\tilde{O} \left( \frac{N}{\sqrt{\mu^*}} \log \left( \frac{1}{\mu^*} \right) \log \left( \frac{n}{p} \right) \log \left( \frac{1}{\epsilon} \right) \right) \approx \log \left( \frac{1}{\epsilon} \right)$$

- Linear in  $N$  and  $\log(1/\epsilon)$

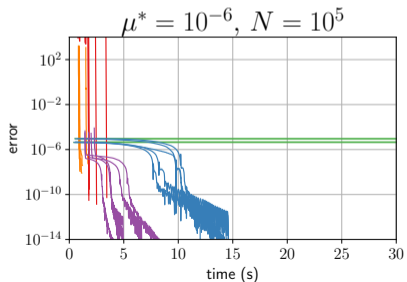
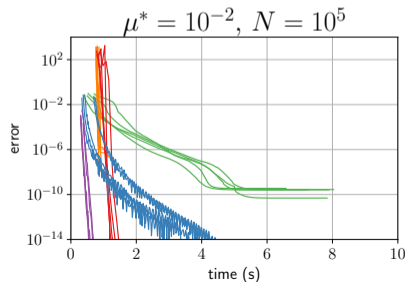
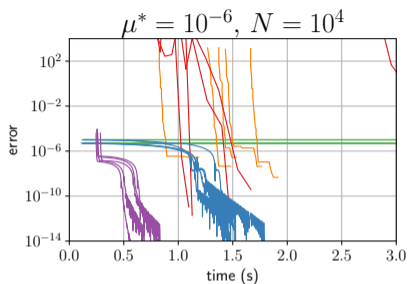
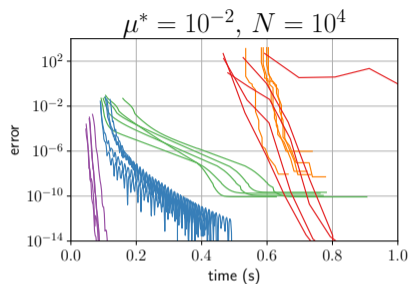
Based on: [\[SIAM J. Optim. under review\]](#)

Related: Carmon and Duchi [2018]

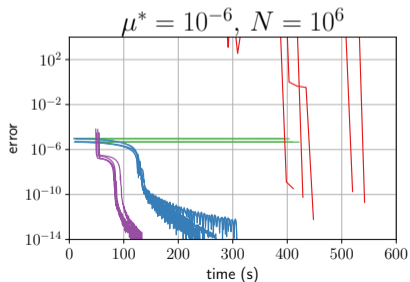
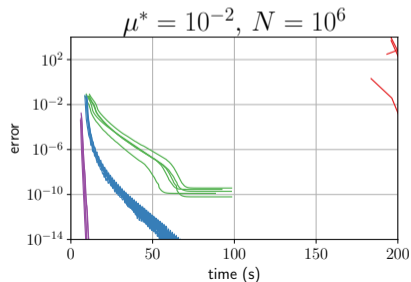
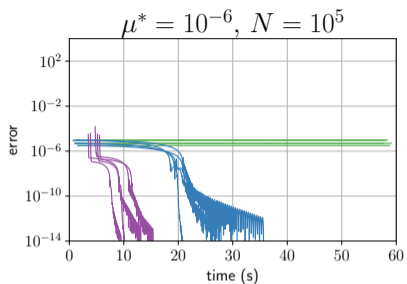
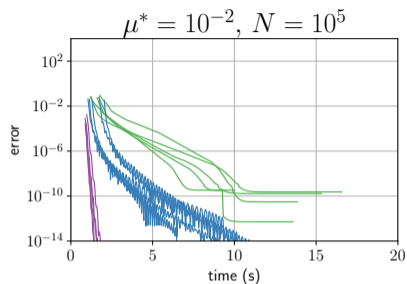
# Numerical experiments, $n = 1,000$



# Numerical experiments, $n = 1,000$

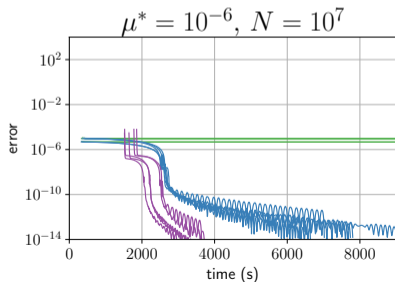
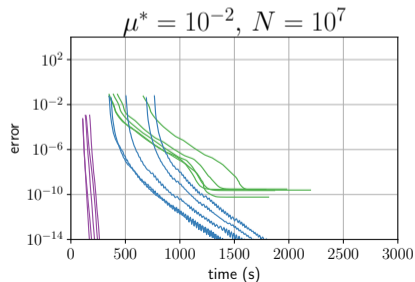
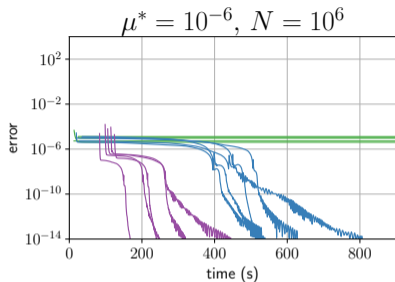
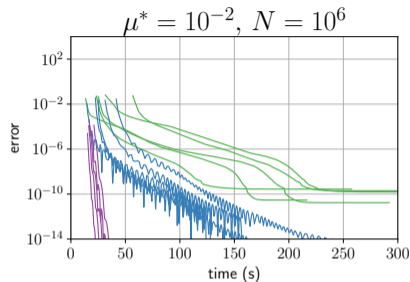


# Numerical experiments, $n = 10,000$





# Numerical experiments, $n = 100,000$



# Efficient algorithms for regular QCQPs

- Regularity can also be defined for **general QCQPs!** (low-rank SDPs)

## Definition

Let  $\gamma^*$  be dual optimizer. Define  $\mu^* := \lambda_{\min} (A_{\text{obj}} + \sum_{i=1}^m \gamma_i^* A_i)$ . QCQP instance is **regular** if  $\mu^* > 0$ .

- $\mu^* > 0 \implies$  objective value exactness
- $\mu^* > 0$  in a number of statistical recovery problems: phase-retrieval, clustering
- Contributions
  - **Conceptual:** If  $\mu^* > 0$ , then can construct strongly convex reformulation in time independent of  $\epsilon$
  - **Algorithmic:** Can solve grad-map efficiently
  - **Practical:** Preliminary implementation and numerical experiments

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Based on: [\[Ongoing\]](#)

# Constructing a strongly convex reformulation of regular QCQP

- Consider a subgradient method for **dual problem**

$$\sup_{\gamma \in \Gamma} \inf_{x \in \mathbb{R}^n} q(\gamma, x)$$

- Iterates  $\gamma^{(1)}, \gamma^{(2)}, \dots$  and bounds  $\|\gamma^{(i)} - \gamma^*\| \leq \rho^{(i)}$

- Then,

$$\text{Opt} = \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in B(\gamma^{(i)}, \rho^{(i)}) \cap \Gamma} q(\gamma, x)$$

- Will stop once  $B(\gamma^{(i)}, \rho^{(i)}) \subseteq \text{int}(\Gamma)$
- Independent of  $\epsilon$ , can be done with  $O(m + n)$  storage

---

Based on: [\[ongoing\]](#)

Related: Ding et al. [2021]

## Preliminary numerical experiments

- 10 synthetic instances:  $n = 5000$ ,  $m = 50$ ,  $\text{density} = 0.001$ ,  $\mu^* = 0.1$
- Primal-dual solver

Section	ncalls	avg time	%tot	avg error
Total	10	1740s	100%	
dual_solve	10	1691s	97.2%	1.15e-02
eigsolve	33.1k	506ms	96.0%	
primal_solve	10	48.7s	2.80%	3.61e-12

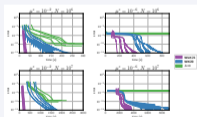
- Splitting Conic Solver (SCS): avg time 18150s

## Summary of Part 2

- Efficient algorithms for the GTRS
- Efficient algorithms for regular QCQPs (low-rank SDPs)
- Algorithms for diagonalizing QCQPs

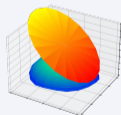
### GTRS

[Math. Prog. 20],  
[SIAM J. Optim. *under review*]



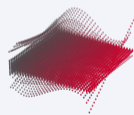
### Regular QCQPs

[Ongoing]



### Diagonalizing QCQPs

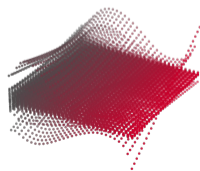
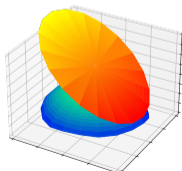
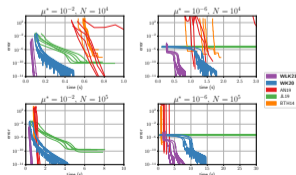
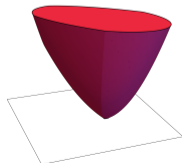
[Math. Prog. *under review*]



### Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

**Some nonconvex problems can be solved efficiently via first-order methods!**



- 1 Preliminaries
- 2 Objective value exactness, convex hull exactness, applications
- 3 Efficient algorithms for structured QCQPs
- 4 Conclusion and future directions**

### Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

- **Solving nonconvex problems accurately**
  - **Completed:** SDPs provide exact reformulations for broad classes of QCQPs!
  - **Future:**
    - Can we understand **approximation quality** systematically within general framework?
    - Can we understand exactness/approximation for **more powerful** convex relaxations?



### Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

- **Solving nonconvex problems efficiently**
  - **Completed:** Some nonconvex problems can be solved efficiently via first-order methods!
  - **Future:**
    - **Exactness  $\approx$  efficiency?**
    - Can we develop efficient algorithms for semidefinite programs with **low-rank solutions**
    - Can we **approximate “expensive” tools** (e.g., SDPs) with **cheap tools** (e.g., linear programs, second-order cone programs)

# Summary of my research

## Exactness

[IPCO 20], [Math. Prog. 21],  
[Math. Prog. *under review*]



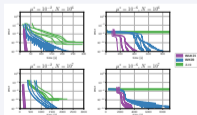
## ROG Cones

[Tut. Oper. Res. 21],  
[Math. Oper. Res. 21]



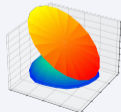
## GTRS

[Math. Prog. 20],  
[SIAM J. Optim. *under review*]



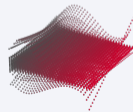
## Regular QCQPs

[Ongoing]



## Diagonalizing QCQPs

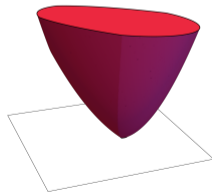
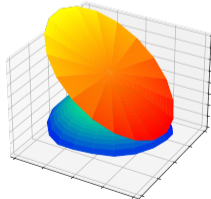
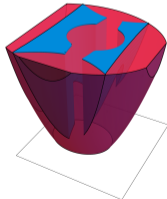
[Math. Prog. *under review*]



## Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

### Thank you! Questions?



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## Definition

Cone  $\mathcal{S} \subseteq \mathbb{S}_+^n$  is rank-one-generated (ROG) if  $\mathcal{S} = \text{conv}(\mathcal{S} \cap \{xx^\top\})$ .

Compare:  $P \subseteq [0, 1]^n$  is integral if  $P = \text{conv}(P \cap \{0, 1\}^n)$

- Given QCQP, if constraints correspond to ROG cone, then objective value exactness and convex hull exactness **regardless of objective function**
- Suppose  $\mathcal{S} = \{X \in \mathbb{S}_+^n : \langle M, X \rangle \leq 0, \forall M \in \mathcal{M}\}$

## Goal

What properties of  $\mathcal{M} = \{M_1, \dots, M_k\}$  imply  $\mathcal{S}$  is ROG?

---

Based on: [\[Math. Oper. Res. 21\]](#), [\[Tut. Oper. Res. 21\]](#)

### Theorem (Sufficient conditions)

$\mathcal{S}$  is ROG if

- for all  $i \neq j$ , there exists  $(\alpha, \beta) \neq (0, 0)$  such that  $\alpha M_i + \beta M_j \succeq 0$ , or
- there exists  $a \in \mathbb{R}^n$  such that  $M_i = ab_i^\top + b_i a^\top$ .

### Theorem (Characterization of ROG for $|\mathcal{M}| = 2$ )

Suppose  $\mathcal{M} = \{M_1, M_2\}$ . Then sufficient condition above is also necessary.

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Based on: [\[Math. Oper. Res. 21\]](#), [\[Tut. Oper. Res. 21\]](#)



## Definition

$\{A_i\} \subseteq \mathbb{S}^n$  is **simultaneously diagonalizable via congruence (SDC)** if there exists invertible  $P \in \mathbb{R}^{n \times n}$  such that  $P^\top A_i P$  is diagonal  $\forall i$ .

- Nice property because: SDP relaxation of diagonal QCQP is SOCP (faster),  $\Gamma$  is polyhedral (better understanding of exactness)

## Goal

Most sets of matrices are not SDC, can we find other computationally variants of SDC and understand such properties?

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Based on: [\[Math. Prog. under review\]](#)

## Definition

$\{A_i\} \subseteq \mathbb{S}^n$  is **almost SDC (ASDC)** if for all  $\epsilon > 0$ , there exists  $\|A'_i - A_i\| \leq \epsilon$  such that  $\{A'_i\}$  is SDC.

- “Limit of SDC sets”

## Definition

$\{A_i\} \subseteq \mathbb{S}^n$  is  **$d$ -restricted SDC ( $d$ -RSDC)** if there exists  $A'_i = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix} \in \mathbb{S}^{n+d}$  such that  $\{A'_i\}$  is SDC.

- “Restriction of SDC sets”

---

Based on: [\[Math. Prog. under review\]](#)

### Theorem

Let  $\{A, B\} \subseteq \mathbb{S}^n$  and suppose  $A$  invertible. Then  $\{A, B\}$  is ASDC if and only if  $A^{-1}B$  has real spectrum. (+ construction)

### Theorem

Let  $\{A, B\} \subseteq \mathbb{S}^n$ . If  $\{A, B\}$  is singular, then it is ASDC. (+ construction)

### Theorem

Let  $\{A, B, C\} \subseteq \mathbb{S}^n$  and suppose  $A$  invertible. Then  $\{A, B, C\}$  is ASDC if and only if  $\{A^{-1}B, A^{-1}C\}$  commute and have real spectrum. (+ construction)

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Based on: [\[Math. Prog. under review\]](#)

### Theorem

Let  $\{A, B\} \subseteq \mathbb{S}^n$ . If  $A$  is invertible and  $A^{-1}B$  has simple eigenvalues, then  $\{A, B\}$  is 1-RSDC. (+ construction)

- Condition holds generically

---

Based on: [\[Math. Prog. under review\]](#)

## “Fuzzy” spectral partitioning

- Connected graph  $G = (V, E)$
- Vertex masses  $\mu : V \rightarrow \mathbb{R}_{++}$  and edge weights  $\kappa : E \rightarrow \mathbb{R}_{++}$
- Laplacian  $L = D - A$  w.r.t.  $\kappa$

### Theorem (Cheeger's inequality)

If  $\mu_v = d_v$ , then 
$$\frac{\Phi^2}{2} \leq \lambda_2(L, M) \leq 2\Phi$$

- $\lambda_2(L, M)$  is first nontrivial generalized eigenvalue
- $\Phi$  is sparsest cut

---

Based on: [APPROX 19]

# “Fuzzy” spectral partitioning

- We define “Fuzzy cuts”

## Definition

$$\Psi \approx \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A, B)}{\min(\mu(A), \mu(B))}, A, B \neq \emptyset, A \cap B = \emptyset \right\}$$

- $\Phi$  must partition,  $\Psi$  may leave out.  $\Psi = \Phi$  if  $A, B$  is a partition.

## Theorem

$$\frac{\Psi}{4} \leq \lambda_2(L, M) \leq \Psi$$

---

Based on: [APPROX 19]

- $k$ -means clustering:  $\{x_1, \dots, x_n\} \subseteq \mathbb{R}^d$
- Suppose there exist true clustering that is unique optimum even if for all  $i$ ,  $x_i \mapsto x'_i \in B(x_i, \epsilon)$

### Theorem

**Two clusters.** There exists  $c \geq 1$  such that for any fixed  $\epsilon > 0$ , we can recover true clustering in time  $d \cdot n^{O(\epsilon^{-c})}$ .

Additional results for  $\geq 3$  clusters given an additional “separation” assumption