Accurately and efficiently solving structured nonconvex optimization problems

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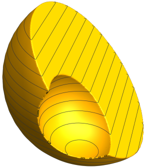
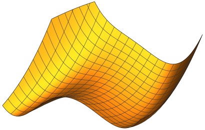
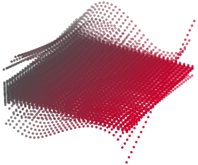
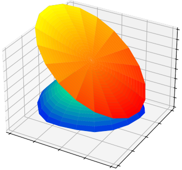
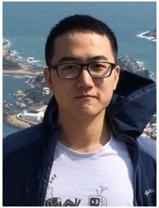
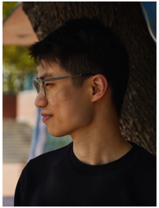
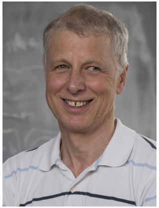
    

These slides are publicly available at `cs.cmu.edu/~alw1`

Collaborators

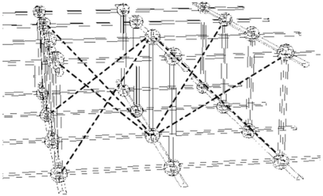
   

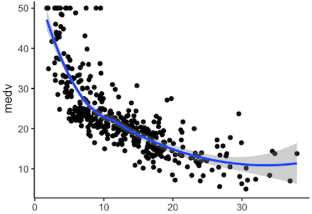
Carnegie Mellon University (OR, Math, CS), Northwestern University, Fudan University, Peking University

Convex optimization

- Convex optimization is influential in many different fields



Engineering
Controller stability, power allocation, truss design, +



Statistics
(Linear) Regression, parameter estimation, +

Finance
Portfolio optimization, risk analysis, +

- Convex optimization is accurate and efficient

Convex optimization, meet nonconvex problems

- Unfortunately, many practical optimization problems are nonconvex
- Example: Low-rank matrix completion (**Netflix problem**)

Movies

Users
$$\begin{pmatrix} 5 & 4 & ? & ? & 4 \\ 3 & ? & ? & 3 & ? \\ ? & 2 & 4 & 1 & 1 \\ ? & 3 & ? & ? & 4 \end{pmatrix}$$

$$\min_{X \in \mathbb{R}^{n \times k}} \{\text{rank}(X) : X \text{ agrees with revealed entries}\}$$

- Rank constraints, binary constraints, sparsity constraints
- Generally hard, **but not always!**
- Some nonconvex problems can be solved using convex optimization

Research goal

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Completed work**:
  - Nonconvex problems: quadratically constrained quadratic programs (QCQPs)
  - Convex relaxations: semidefinite programs (SDPs)

Today's questions

Understand **structures within QCQPs** that enable us to solve them exactly and efficiently **using SDPs**

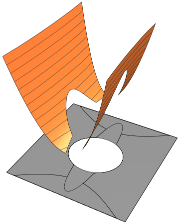
Today's outline

- **Preliminaries**
  QCQPs and their applications, the SDP relaxation
- **Understand structures within QCQPs that enable us to solve them. . .**
  - **exactly** [IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*]
    Objective value, convex hull exactness, applications
  - **efficiently** [Math. Prog. 20], [SIAM J. Optim. *in prep.*], [Ongoing]
    The generalized trust-region subproblem and regular QCQPs
- **Conclusion and future directions**

1. Preliminaries
2. Objective value exactness, convex hull exactness, applications
3. Efficient algorithms for structured QCQPs
4. Conclusion and future directions

Quadratically constrained quadratic programs (QCQPs)

- $q_{\mathsf{obj}}, q_1, \dots, q_m : \mathbb{R}^n \to \mathbb{R}$ quadratic (possibly nonconvex!)



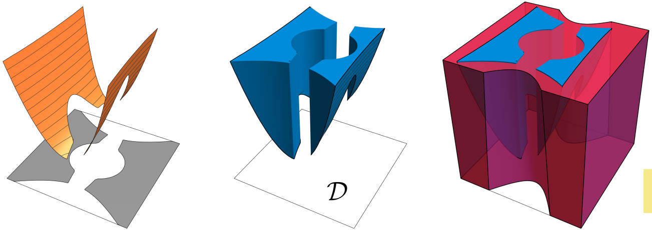
$$q_i(x) = x^\intercal A_i x + 2b_i^\intercal x + c_i$$

$$\mathrm{Opt} := \inf_{x \in \mathbb{R}^n} \left\{ q_{\mathsf{obj}}(x) : \; q_i(x) \leq 0, \; \forall i \in [m] \right\}$$

- Highly expressive:
  - MAX-CUT, MAX-CLIQUE, pooling, truss design, facility location, production planning
  - binary program $x_1(1 - x_1) = 0$
  - polynomial optimization problems $x_1 x_2 = z_{12}$
- NP-hard in general

The QCQP epigraph

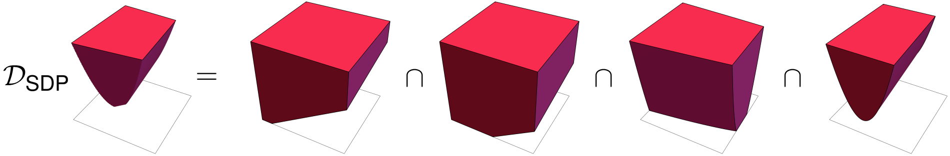
- QCQP epigraph $$\mathcal{D} := \left\{ (x,t) \in \mathbb{R}^{n+1} : \begin{array}{l} q_{\text{obj}}(x) \leq t \\ q_i(x) \leq 0,\ \forall i \in [m] \end{array} \right\}$$



$q(\gamma', x) \leq t$

- How can we derive ~~convex~~ relaxations of $\mathcal{D}$?
- If $\gamma \in \mathbb{R}^m_+$, then $$\forall (x,t) \in \mathcal{D}, \quad \underbrace{q_{\text{obj}}(x) + \sum_{i=1}^{m} \gamma_i q_i(x)}_{=:\, q(\gamma, x)} \leq t$$

- SDP relaxation = impose all convex aggregated inequalities!



- Formally,

$$\Gamma := \left\{\gamma \in \mathbb{R}^m_+ : q(\gamma, x) \text{ is convex in } x\right\} = \left\{\gamma \in \mathbb{R}^m_+ : A_{\text{obj}} + \sum_{i=1}^m \gamma_i A_i \succeq 0\right\}$$

$$\mathcal{D}_{\text{SDP}} := \bigcap_{\gamma\in\Gamma} \left\{(x,t) : q(\gamma, x) \leq t\right\} = \left\{(x,t) \in \mathbb{R}^{n+1} : \sup_{\gamma\in\Gamma} q(\gamma, x) \leq t\right\}$$

$$\text{Opt}_{\text{SDP}} := \inf_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q(\gamma, x)$$

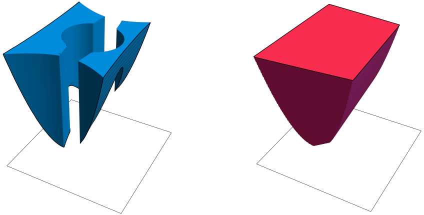
The usual SDP relaxation

$$
\begin{aligned}
&\inf_{x\in\mathbb{R}^n} \{q_{\text{obj}}(x) : q_i(x) \le 0,\ \forall i\in[m]\}\\
&= \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X\rangle + 2b_{\text{obj}}^\top x + c_{\text{obj}} : \begin{array}{l} X = xx^\top \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0,\ \forall i\in[m] \end{array} \right\}\\
&\ge \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X\rangle + 2b_{\text{obj}}^\top x + c_{\text{obj}} : \begin{array}{l} X - xx^\top \succeq 0 \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0,\ \forall i\in[m] \end{array} \right\}\\
&= \inf_{x\in\mathbb{R}^n} \inf_{X\in\mathbb{S}^n} \dots\\
&= \inf_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q(\gamma, x)
\end{aligned}
$$

Preliminaries recap

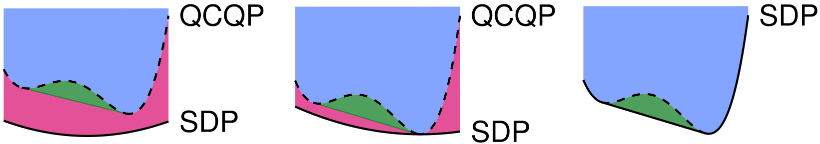
- Main objects of interest

	nonconvex QCQP	convex SDP
Optimum value	Opt	$\mathrm{Opt}_{\mathsf{SDP}}$
Epigraph	$\mathcal{D}$	$\mathcal{D}_{\mathsf{SDP}}$



- Useful for analysis:
  - $q(\gamma, x) =$ Lagrangian function
  - $\Gamma =$ aggregation weights giving convex $q(\gamma, x)$

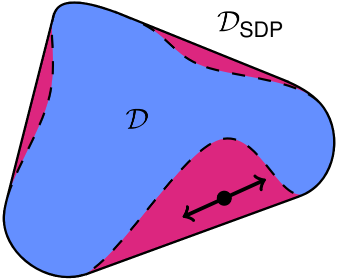
- What does exactness mean?
  - Objective value exactness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$
  - Convex hull exactness: $\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}}$ $\leftarrow$ convexification of substructures



Obj. val. ex. ✗	Obj. val. ex. ✓	Obj. val. ex. ✓
Conv. hull ex. ✗	Conv. hull ex. ✗	Conv. hull ex. ✓

Convex hull exactness

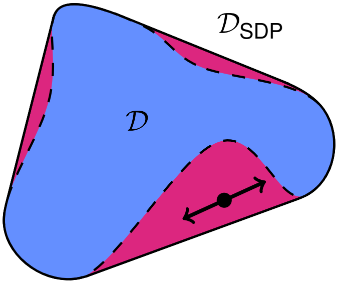
- $\mathrm{conv}(\mathcal{D}) \overset{?}{=} \mathcal{D}_{\mathsf{SDP}}$



$\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}} \iff$ "Given any point in $\mathcal{D}_{\mathsf{SDP}} \setminus \mathcal{D}$, exists direction such that can move forward and backward inside $\mathcal{D}_{\mathsf{SDP}}$"

Sufficient conditions for exactness

- Can carry out this idea for QCQPs!
- Leads to sufficient conditions based on abstract properties



Theorem ([Math. Prog. 21])

Suppose Γ polyhedral. If for every semidefinite face $\mathcal{F} \trianglelefteq \Gamma$,

$$\operatorname{aff}\left(\operatorname{Proj}_{\mathcal{V}(\mathcal{F})}\{b(\gamma) : \gamma \in \mathcal{F}\}\right) \neq \mathcal{V}(\mathcal{F}),$$

then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}}$.

Theorem ([Math. Prog. *under review*])

If for every $(x,t) \in \mathcal{D}_{\mathsf{SDP}} \setminus \mathcal{D}$,

$$\left\{(x',t') \in \mathbb{R}^{n+1} : \begin{array}{l} x' \in \ker(A(f)) \\ \langle A(\eta)x + b(\eta), x'\rangle - t' = 0,\ \forall (1,\eta) \in \mathcal{G}^\perp \end{array}\right\} \neq \{0\},$$

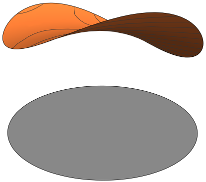
then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}}$.

Based on: **[IPCO 19]**, **[Math. Prog. 21]**, **[Math. Prog. *under review*]**

Example: the trust-region subproblem

- Convex hull exactness in the case of single ball constraint

$$\mathrm{Opt} = \inf_{x\in\mathbb{R}^n} \left\{ q_{\mathsf{obj}}(x) :\ \|x\|^2 \le 1 \right\}$$



- **Applications**:
  - Nonlinear minimization (trust-region methods), combinatorial optimization, robust optimization

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Based on: **[IPCO 19]**, **[Math. Prog. 20]**
Related: Yakubovich [1971], Yıldıran [2009], Ho-Nguyen and Kılınç-Karzan [2017]

Example: QCQPs with symmetry

- Convex hull exactness in the case of "highly symmetric" QCQPs
- Suppose $A_{\text{obj}} = I_k \otimes \mathbb{A}_{\text{obj}}$, $A_i = I_k \otimes \mathbb{A}_i$ for all $i \in [m]$

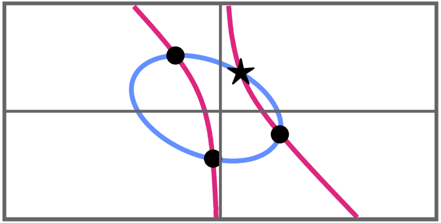
$$I_k \otimes \mathbb{A} = \begin{pmatrix} \mathbb{A} & & & \\ & \mathbb{A} & & \\ & & \ddots & \\ & & & \mathbb{A} \end{pmatrix}$$

  and $k \geq m$
- **Applications**:
  - Robust least squares, sphere packing, QCQPs with spherical constraints, orthogonal Procrustes problem

Based on: **[Math. Prog. *under review*]**
Related: Beck [2007], Beck et al. [2012]

Example: Random underconstrained quadratic systems

- Obj. val. exactness in the case of random underconstrained quadratic systems
- Solve $$\inf_{x\in\mathbb{R}^n} \left\{ \|x\|^2 : q_i(x) = 0,\ \forall i \in [m] \right\}$$



- Fix m, let $n \to \infty$, if data generated “as Gaussians”, then objective value exactness w.p. $1 - o(1)$

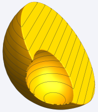
Based on: **[Math. Prog. *under review*]**
Related: Burer and Ye [2019], Locatelli [2020]

Summary of Part 1

- Sufficient conditions for convex hull exactness
- **Necessary and sufficient** if Γ is polyhedral (dual facially exposed)
- Sufficient conditions for objective value exactness
- Rank-one-generated (ROG) cones: strengthening of convex hull exactness
- **Applications**:
  - Diagonal QCQPs with sign-definite linear terms, semi-random QCQPs, ratios of quadratic functions

Exactness

[IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*]



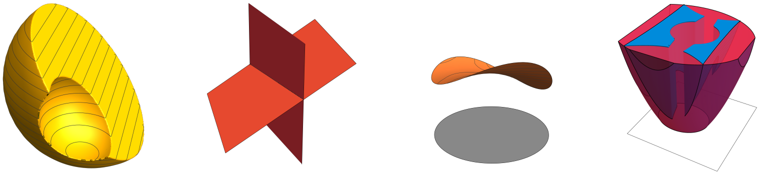
ROG Cones

[Tut. Oper. Res. 21], [Math. Oper. Res. 21]

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

SDPs provide exact reformulations for broad classes of QCQPs!



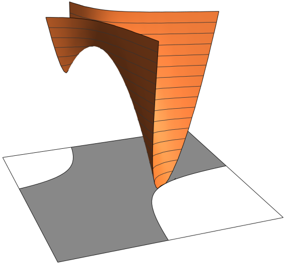
Revisiting the SDP relaxation

- Usual SDP relaxation in $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$ $\implies \approx n^2$ variables
- Our view:
$$\text{Opt}_{\text{SDP}} := \inf_{x \in \mathbb{R}^n} \left(\sup_{\gamma \in \Gamma} q(\gamma, x) \right)$$
is a minimization problem in the original space $\implies n$ variables

The generalized trust-region subproblem (GTRS)

- Special setting with single constraint ($\leq$ or $=$)

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \leq 0\}$$



- **TRS Applications**:
  - nonlinear programming (trust-region methods), combinatorial optimization, robust optimization
- **GTRS Applications**:
  - minimizing quartics of the form $q(x, p(x))$

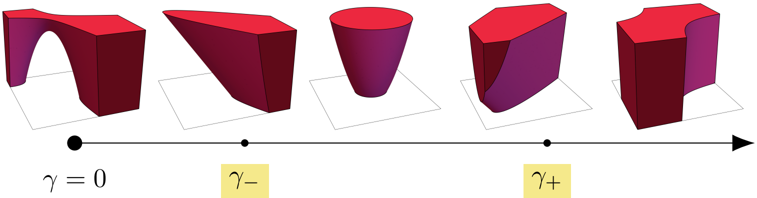
$$\inf_{x \in \mathbb{R}^n, \alpha} \{q(x, \alpha) : \alpha = p(x)\}$$

(source localization, constrained rank-one approximation), iterative QCQP solvers

- Convex hull exactness holds $\quad \mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}} = \left\{ (x,t) :\ \sup_{\gamma\in\Gamma} q(\gamma,x) \le t \right\}$

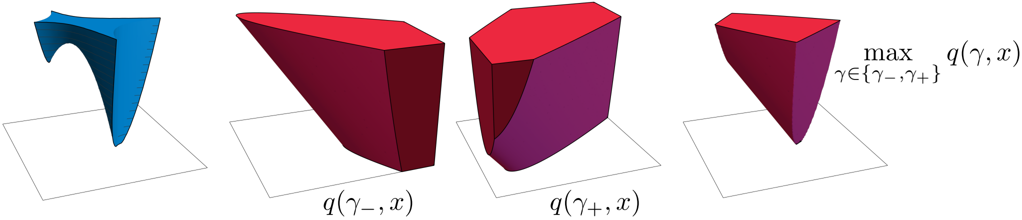
- Recall $\quad \Gamma = \{\gamma \in \mathbb{R}_+ :\ q(\gamma,x) \text{ is convex in } x\}$

$q(\gamma, x)$



Based on: [Math. Prog. 20]
Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

- $\Gamma = [\gamma_-, \gamma_+] \implies \text{Opt} = \text{Opt}_{\mathsf{SDP}} = \inf_{x\in\mathbb{R}^n} \max_{\gamma\in\{\gamma_-,\gamma_+\}} q(\gamma, x)$



- **Algorithmic idea**
  - Compute γ_- and γ_+ to some accuracy
  - Apply accelerated gradient descent

$$\implies \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \frac{1}{\sqrt{\epsilon}}$$

Based on: **[Math. Prog. 20]**
Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

Is this running time optimal?

- GTRS:

$$\inf_{x\in\mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \leq 0\} \qquad \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right)$$

- Minimum eigenvalue:

$$\inf_{x\in\mathbb{R}^n} \left\{x^\intercal A x : \|x\|^2 = 1\right\} \qquad O\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right)\right)$$

- Smooth convex quadratic:

$$\inf_{x\in\mathbb{R}^n} q_{\text{obj}}(x) \qquad O\left(\frac{N}{\sqrt{\epsilon}}\right)$$

- But, these are hardest problems within the GTRS!
- Can do better when regularity $\mu^* > 0$

Based on: **[Math. Prog. 20]**
Related: Kuczynski and Wozniakowski [1992], Nesterov [2018]

Linear convergence for regular GTRS

- $\mu^* > 0$ holds for most GTRS
- Dual problem

$$\mathrm{Opt}_{\mathsf{SDP}} := \inf_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q(\gamma, x) = \sup_{\gamma\in\Gamma} \inf_{x\in\mathbb{R}^n} q(\gamma, x)$$

Definition

Let γ^* be dual optimizer. Define $\mu^* := \lambda_{\min}(A_{\mathsf{obj}} + \gamma^* A_1)$. GTRS instance is regular if $\mu^* > 0$.

- Minimum eigenvalue problem is not regular
- Some instances of smooth quadratic minimization are not regular

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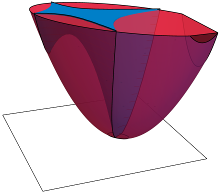
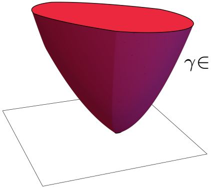
Based on: **[SIAM J. Optim. *in prep.*]**
Related: Carmon and Duchi [2018]

Linear convergence for regular GTRS

- Suppose $\mu^* > 0$





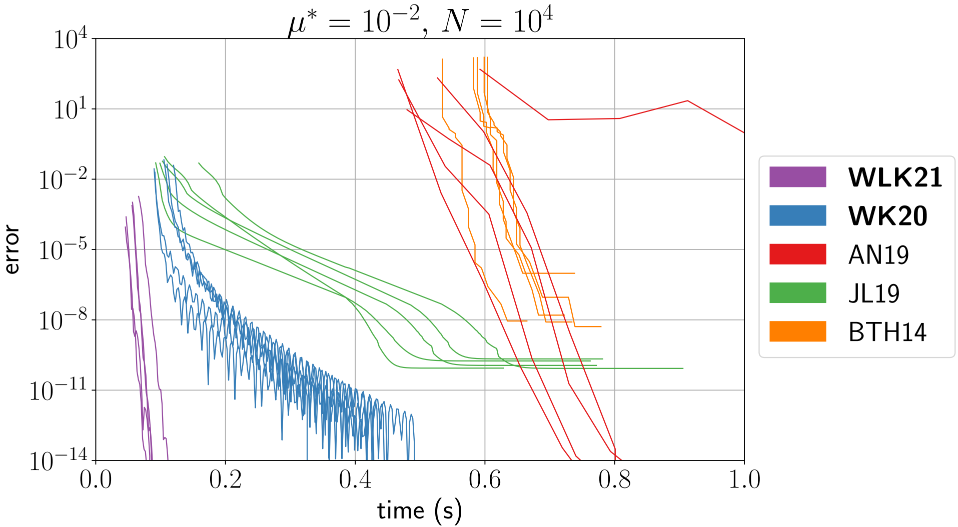
- Suppose $\gamma^* \in [\tilde{\gamma}_-, \tilde{\gamma}_+] \subseteq \Gamma \implies \mathrm{Opt} = \inf_{x\in\mathbb{R}^n} \max_{\gamma\in\{\tilde{\gamma}_-,\tilde{\gamma}_+\}} q(\gamma, x)$
- Suffices to estimate γ^* roughly and can exploit strong convexity

$$\tilde{O}\left(\frac{N}{\sqrt{\mu^*}} \log\left(\frac{1}{\mu^*}\right) \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \log\left(\frac{1}{\epsilon}\right)$$

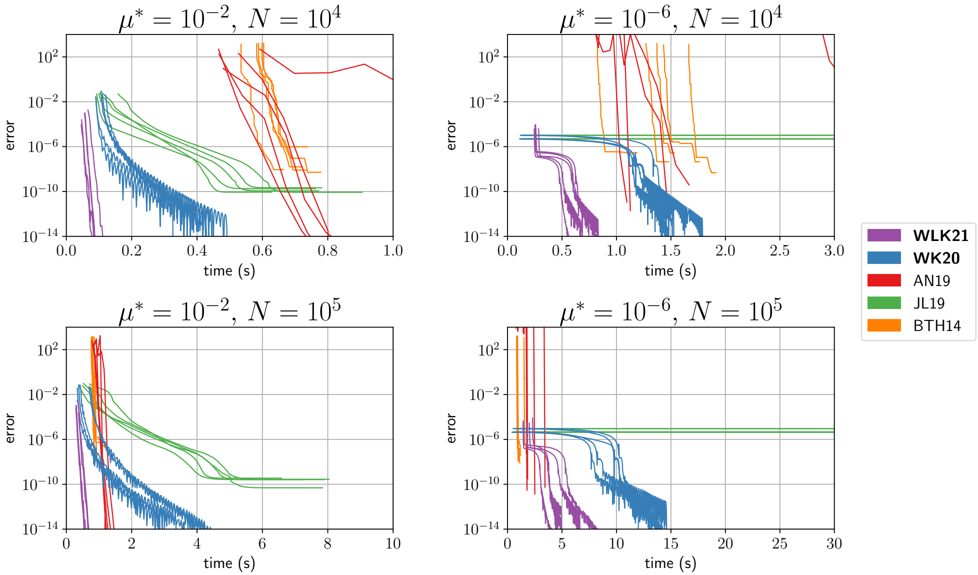
- Linear in N and $\log(1/\epsilon)$

Based on: **[SIAM J. Optim. *in prep.*]**
Related: Carmon and Duchi [2018]

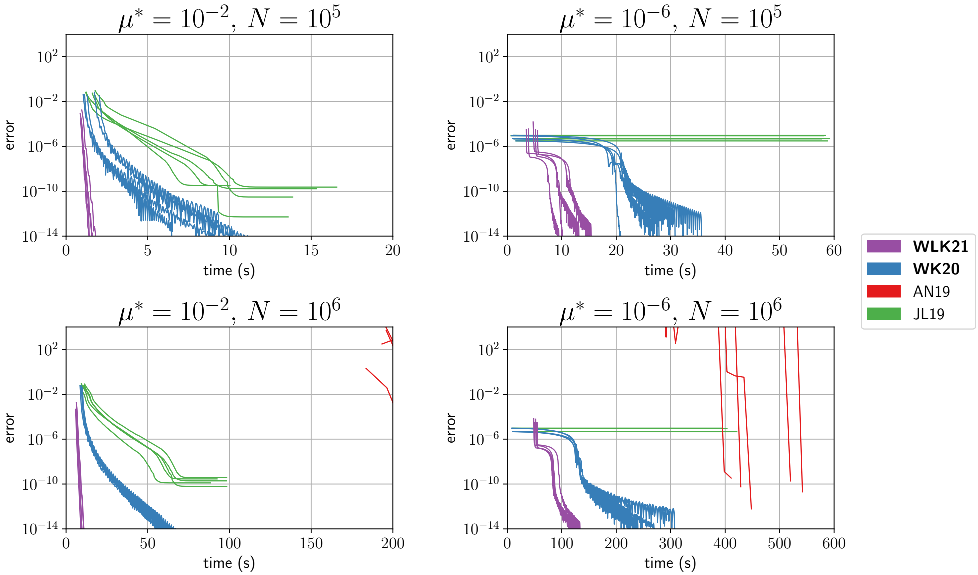
Numerical experiments, n = 1,000



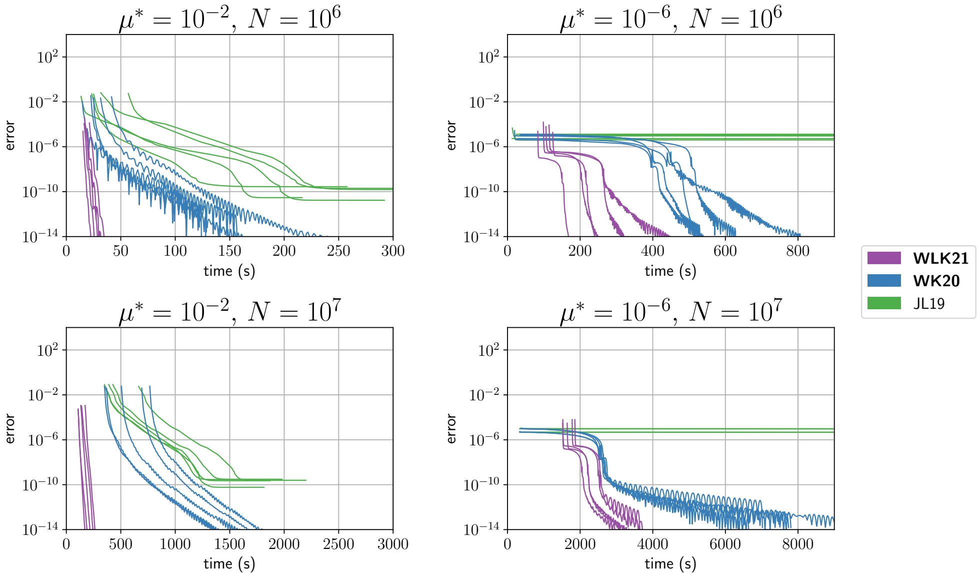
Numerical experiments, n = 1,000



Numerical experiments, n = 10,000



Numerical experiments, n = 100,000



Efficient algorithms for regular QCQPs

- Regularity can also be defined for general QCQPs!

Definition

Let γ^* be dual optimizer. Define $\mu^* := \lambda_{\min}\left(A_{\text{obj}} + \sum_{i=1}^m \gamma_i^* A_i\right)$. QCQP instance is regular if $\mu^* > 0$.

- $\mu^* > 0 \implies$ objective value exactness
- $\mu^* > 0$ in a number of statistical recovery problems
- If $\gamma^* \in B(\tilde{\gamma}, \delta) \subseteq \Gamma$

$$\text{Opt} = \text{Opt}_{\text{SDP}} = \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x) = \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in B(\tilde{\gamma}, \delta)} q(\gamma, x)$$

- **Applications**:
  - TRS, GTRS, random QCQPs, phase-retrieval, clustering

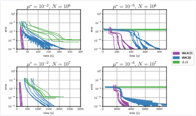
Based on: **[Ongoing]**

Summary of Part 2

- Efficient algorithms for the GTRS
- Efficient algorithms for regular QCQPs
- Algorithms for diagonalizing QCQPs

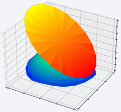
GTRS

[Math. Prog. 20],
[SIAM J. Optim. *in prep.*]



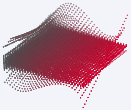
Regular QCQPs

[Ongoing]



Diagonalizing QCQPs

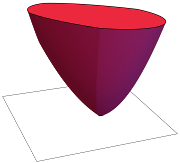
[Math. Prog. *under review*]

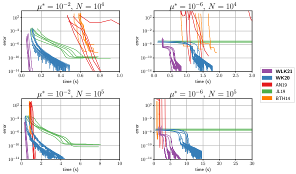


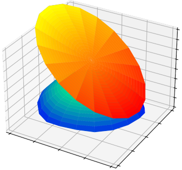
Long-term research goal

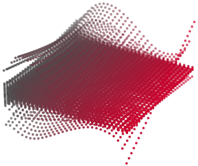
Understand **structures within nonconvex problems** that enable us to solve them “well” using **convex optimization**

Some nonconvex problems can be solved efficiently via first-order methods!









1. Preliminaries
2. Objective value exactness, convex hull exactness, applications
3. Efficient algorithms for structured QCQPs
4. Conclusion and future directions

Future work

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Solving nonconvex problems accurately**
  - **Completed**: SDPs provide exact reformulations for broad classes of QCQPs!
  - **Future**:
    - Can we understand approximation quality systematically within general framework?
    - Can we understand exactness/approximation for more powerful convex relaxations?

Future work

Long-term research goal

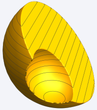
Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Solving nonconvex problems efficiently**
  - **Completed**: Some nonconvex problems can be solved efficiently via first-order methods!
  - **Future**:
    - Exactness $\approx$ efficiency?
    - Can we develop efficient algorithms for semidefinite programs with low-rank solutions
    - Can we approximate "expensive" tools (e.g., SDPs) with cheap tools (e.g., linear programs, second-order cone programs)

Summary of my research

Exactness

[IPCO 20], [Math. Prog. 21],
[Math. Prog. *under review*]

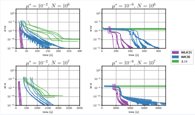


ROG Cones

[Tut. Oper. Res. 21],
[Math. Oper. Res. 21]

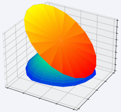
GTRS

[Math. Prog. 20],
[SIAM J. Optim. *in prep.*]



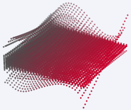
Regular QCQPs

[Ongoing]



Diagonalizing QCQPs

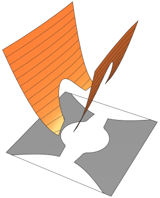
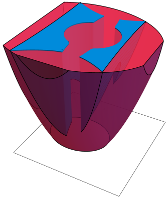
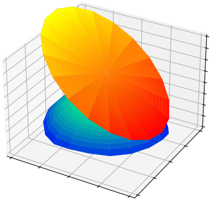
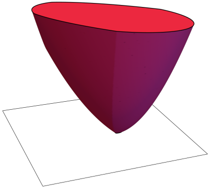
[Math. Prog. *under review*]



Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

Thank you! Questions?

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Definition

Cone $\mathcal{S} \subseteq \mathbb{S}^n_+$ is rank-one-generated (ROG) if $\mathcal{S} = \operatorname{conv}\left(\mathcal{S} \cap \left\{xx^\top\right\}\right)$.

Compare: $P \subseteq [0,1]^n$ is integral if $P = \operatorname{conv}(P \cap \{0,1\}^n)$

- Given QCQP, if constraints correspond to ROG cone, then objective value exactness and convex hull exactness regardless of objective function
- Suppose $\mathcal{S} = \left\{X \in \mathbb{S}^n_+ : \langle M, X\rangle \leq 0,\ \forall M \in \mathcal{M}\right\}$

Goal

What properties of $\mathcal{M} = \{M_1, \ldots, M_k\}$ imply $\mathcal{S}$ is ROG?

Based on: [Math. Oper. Res. 21], [Tut. Oper. Res. 21]

Rank-one-generated cones

Theorem (Sufficient conditions)

$\mathcal{S}$ is ROG if

- for all $i \neq j$, there exists $(\alpha, \beta) \neq (0,0)$ such that $\alpha M_i + \beta M_j \succeq 0$, or
- there exists $a \in \mathbb{R}^n$ such that $M_i = ab_i^\top + b_i a^\top$.

Theorem (Characterization of ROG for $|\mathcal{M}| = 2$)

Suppose $\mathcal{M} = \{M_1, M_2\}$. Then sufficient condition above is also necessary.

Based on: [Math. Oper. Res. 21], [Tut. Oper. Res. 21]

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is simultaneously diagonalizable via congruence (SDC) if there exists invertible $P \in \mathbb{R}^{n\times n}$ such that $P^\top A_i P$ is diagonal $\forall i$.

- Nice property because: SDP relaxation of diagonal QCQP is SOCP (faster), Γ is polyhedral (better understanding of exactness)

Goal

Most sets of matrices are not SDC, can we find other computationally variants of SDC and understand such properties?

Based on: **[Math. Prog. *under review*]**

Diagonalizing QCQPs: variants of SDC

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is almost SDC (ASDC) if for all $\epsilon > 0$, there exists $\|A_i' - A_i\| \leq \epsilon$ such that $\{A_i'\}$ is SDC.

- "Limit of SDC sets"

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is d-restricted SDC (d-RSDC) if there exists $A_i' = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix} \in \mathbb{S}^{n+d}$ such that $\{A_i'\}$ is SDC.

- "Restriction of SDC sets"

Based on: [Math. Prog. *under review*]

Diagonalizing QCQPs: ASDC

Theorem

Let $\{A, B\} \subseteq \mathbb{S}^n$ and suppose A invertible. Then $\{A, B\}$ is ASDC if and only if $A^{-1}B$ has real spectrum. (+ construction)

Theorem

Let $\{A, B\} \subseteq \mathbb{S}^n$. If $\{A, B\}$ is singular, then it is ASDC. (+ construction)

Theorem

Let $\{A, B, C\} \subseteq \mathbb{S}^n$ and suppose A invertible. Then $\{A, B, C\}$ is ASDC if and only if $\{A^{-1}B, A^{-1}C\}$ commute and have real spectrum. (+ construction)

Based on: **[Math. Prog. *under review*]**

Theorem

Let $\{A, B\} \subseteq \mathbb{S}^n$. If A is invertible and $A^{-1}B$ has simple eigenvalues, then $\{A, B\}$ is 1-RSDC. (+ construction)

- Condition holds generically

Based on: **[Math. Prog. *under review*]**

"Fuzzy" spectral partitioning

- Connected graph $G = (V, E)$
- Vertex masses $\mu : V \to \mathbb{R}_{++}$ and edge weights $\kappa : E \to \mathbb{R}_{++}$
- Laplacian $L = D - A$ w.r.t. κ

Theorem (Cheeger's inequality)

If $\mu_v = d_v$, then

$$\frac{\Phi^2}{2} \leq \lambda_2(L, M) \leq 2\Phi$$

- $\lambda_2(L, M)$ is first nontrivial generalized eigenvalue
- Φ is sparsest cut

Based on: [APPROX 19]

"Fuzzy" spectral partitioning

- We define "Fuzzy cuts"

Definition

$$\Psi \approx \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\min\left(\mu(A), \mu(B)\right)}, \; A, B \neq \varnothing, \; A \cap B = \varnothing \right\}$$

- Φ must partition, Ψ may leave out. $\Psi = \Phi$ if A, B is a partition.

Theorem

$$\frac{\Psi}{4} \leq \lambda_2(L, M) \leq \Psi$$

Based on: [APPROX 19]

- k-means clustering: $\{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$
- Suppose there exist true clustering that is unique optimum even if for all i, $x_i \mapsto x_i' \in B(x_i, \epsilon)$

Theorem

Two clusters. There exists $c \geq 1$ such that for any fixed $\epsilon > 0$, we can recover true clustering in time $d \cdot n^{O(\epsilon^{-c})}$.

Additional results for ≥ 3 clusters given an additional "separation" assumption

Based on: [NeurIPS 17]