Accelerated first-order methods for a class of semidefinite programs

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- Semidefinite program

Related: Beck [2007], Beck et al. [2012], Wang and Kılınç-Karzan [2020]

SDPs and QMPs

- Semidefinite program

$$(\textbf{SDP}) = \inf_{Y \in \mathbb{S}^{n+k}} \left\{ \langle M_0, Y \rangle : \begin{array}{l} \langle M_i, Y \rangle + d_i = 0, \ \forall i \in [m] \\ Y \succeq 0 \end{array} \right\}$$

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- **Sneak peek**: "k-exact" SDPs can be converted to easy instances of QMP

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Structural assumptions on SDP

- $$(\textbf{SDP}) = \inf_{Y \in \mathbb{S}^{n+k}} \left\{ \langle M_0, Y \rangle : \begin{array}{l} \langle M_i, Y \rangle + d_i = 0,\ \forall i \in [m] \\ Y \succeq 0 \end{array} \right\}$$

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- This talk: $W = \mathbb{R}^n \times \{0_k\}$ and $Y^*_{W^\perp} = I_k$

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Motivation

- SDP relaxation of QMP in $X \in \mathbb{R}^{n\times k}$

Related: Ben-Tal and Nemirovski [2001], Beck [2007], Wang and Kılınç-Karzan [2022]

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- SDP relaxation of QMP in $X \in \mathbb{R}^{n\times k}$
- Let $Y(X) := \begin{pmatrix} XX^\intercal & X \\ X^\intercal & I_k \end{pmatrix}$

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Know $Y^*_{W^\perp} = I_k$

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Recap of structural assumptions

- **k-exact SDPs**

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&= \mathop{\arg\min}_{X \in \mathbb{R}^{n\times k}} \left\{ q_0(X) : q_i(X) = 0,\ \forall i \in [m] \right\}
\end{aligned}
$$

- Have reduced SDP $\longrightarrow$ QMP (Easy?)

Lagrangian reformulation

- Take the Lagrangian to get minimax problem without constraints

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- $q(\gamma, X) = \operatorname{tr}(X^\intercal A(\gamma) X) + \ldots$ where

$$A(\gamma) = \text{top-left block of } M(\gamma)$$

Deriving a strongly convex minimax problem

- **Thought experiment**: strong duality + strict complementarity implies

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- Strict complementarity $\longrightarrow$ $A(\gamma^*) \succ 0$

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Theorem (Certificate of strict compl. gives strongly conv. reform.)

Suppose $\gamma^* \in \mathcal{C} \subseteq \mathbb{R}^m$ and $A(\gamma) \succ 0$ for all $\gamma \in \mathcal{C}$, then

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  - Construct $\mathcal{C}$

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$$X^* = \arg\min_{X \in \mathbb{R}^{n\times k}} \max_{\gamma \in \mathcal{C}} q(\gamma, X)$$

- **Algorithm**:
  - Construct $\mathcal{C}$
  - Solve strongly convex quadratic matrix minimax problem (QMMP)

Algorithms: CautiousAGD

- How to solve strongly convex QMMP?

$$\arg\min_{X \in \mathbb{R}^{n \times k}} \max_{\gamma \in \mathcal{C}} q(\gamma, X)$$

Related: Nesterov [2005], Devolder et al. [2013, 2014], Nesterov [2018]

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Theorem (CautiousAGD)

CautiousAGD produces iterates X_t such that

$$\max_{\gamma\in\mathcal{C}} q(\gamma, X_t) \le \min_{X} \max_{\gamma\in\mathcal{C}} q(\gamma, X) + \epsilon$$

after $O\left(\log(\epsilon^{-1})\right)$ iterations, $O(m\epsilon^{-1/2})$ matrix-vector products per iteration

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Related: Nesterov [2005], Devolder et al. [2013, 2014], Nesterov [2018]

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Algorithms: CertSDP

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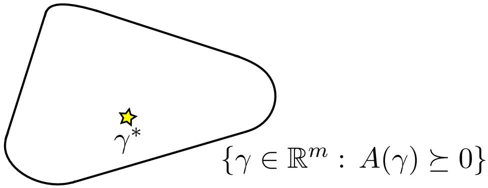

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- How to construct $\mathcal{C}$? $\gamma^* \in \mathcal{C}$, $A(\gamma) \succ 0$ for all $\gamma \in \mathcal{C}$
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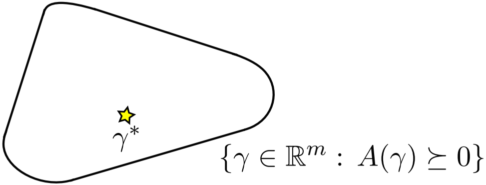

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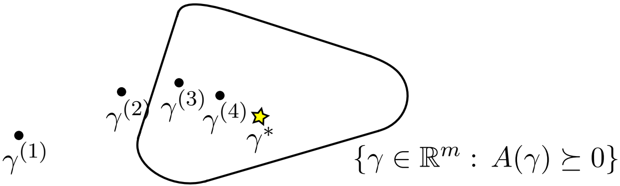

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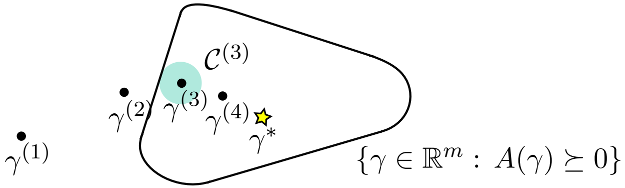

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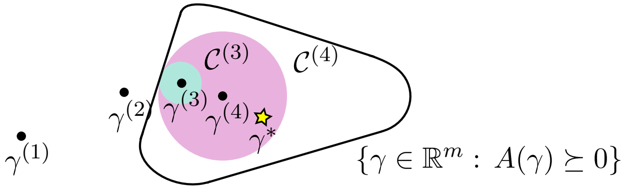

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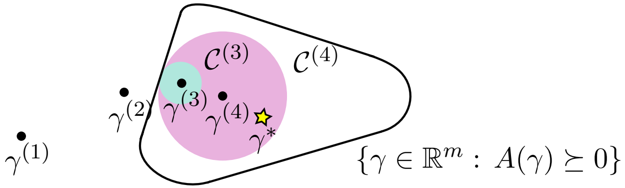

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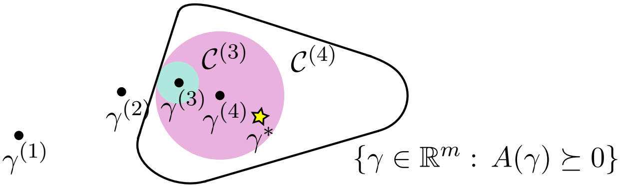

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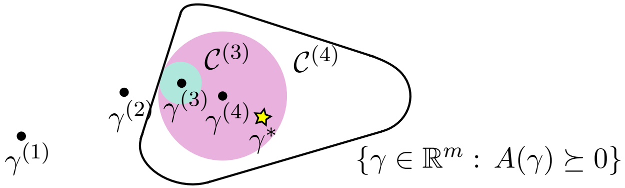

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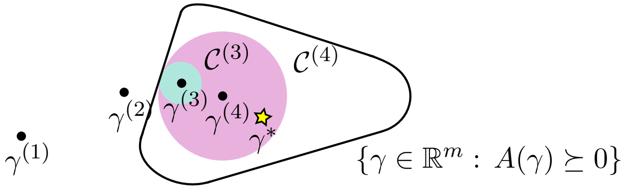

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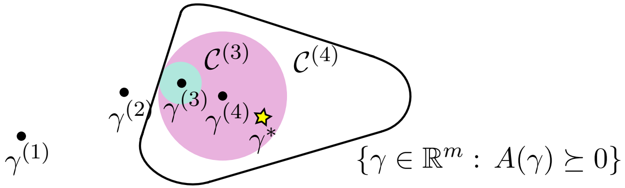
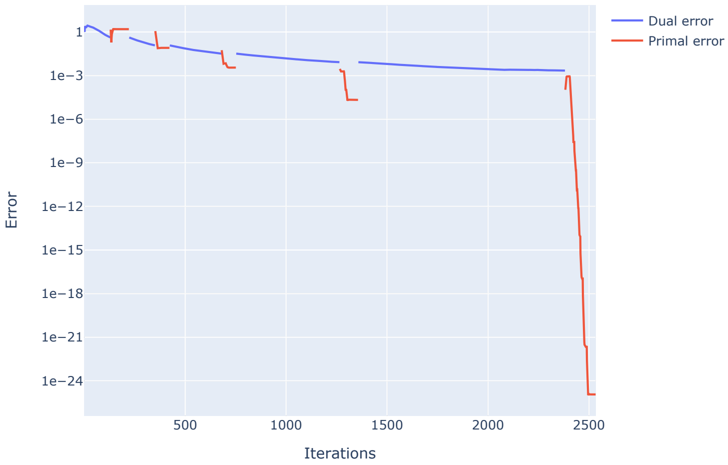


CertSDP convergence behavior



CertSDP guarantees

Theorem (CertSDP)

CertSDP produces iterates X_t such that

$$\left\langle M_0, \begin{pmatrix} X_t X_t^\intercal & X_t \\ X_t^\intercal & I_k \end{pmatrix} \right\rangle \leq \mathrm{Opt}_{\textsf{SDP}} + \epsilon \qquad \left\| \left(\left\langle M_i, \begin{pmatrix} X_t X_t^\intercal & X_t \\ X_t^\intercal & I_k \end{pmatrix} \right\rangle + d_i \right)_i \right\|_2 \leq \epsilon$$

Related: Ding et al. [2021], Yurtsever et al. [2021], Friedlander and Macêdo [2016], Shinde et al. [2021]

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1. What structure? . . . k-exactness
2. How to use k-exactness? . . . Strongly convex reformulation
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Thank you! Questions?

`https://arxiv.org/abs/2206.00224`

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