# Accelerated first-order methods for a class of semidefinite programs

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  - Simply writing down $Y$ requires $O((n+k)^2)$ memory

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- Have reduced SDP $\longrightarrow$ QMP:

$$Y^* = \begin{pmatrix} X^*(X^*)^\intercal & X^* \\ (X^*)^\intercal & I_k \end{pmatrix}$$

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# Lagrangian reformulation

- Take the Lagrangian to get minimax problem without constraints

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$$A(\gamma) = \text{top-left block of } M(\gamma)$$

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Suppose $\gamma^* \in \mathcal{C} \subseteq \mathbb{R}^m$ and $A(\gamma) \succ 0$ for all $\gamma \in \mathcal{C}$, then

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- **Algorithm**:
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  - Solve strongly convex quadratic matrix minimax problem (QMMP)

# Algorithms: CautiousAGD

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Related: Nesterov [2005], Devolder et al. [2013, 2014], Nesterov [2018]

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**Theorem (CautiousAGD)**

CautiousAGD produces iterates $X_t$ such that

$$\max_{\gamma\in\mathcal{C}} q(\gamma, X_t) \le \min_X \max_{\gamma\in\mathcal{C}} q(\gamma, X) + \epsilon$$

after $O\left(\log(\epsilon^{-1})\right)$ iterations, $O(m\epsilon^{-1/2})$ matrix-vector products per iteration

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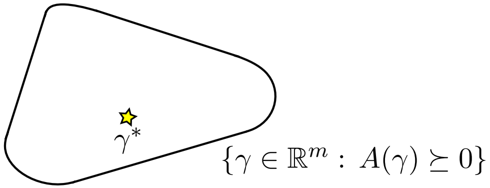

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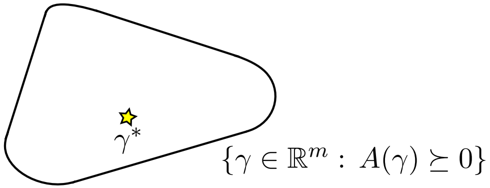

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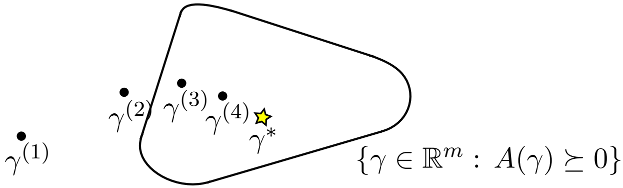

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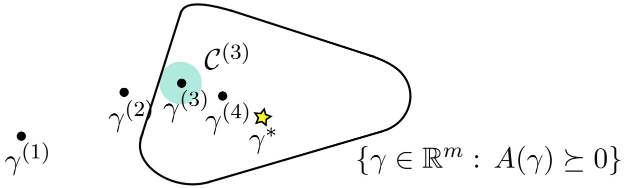

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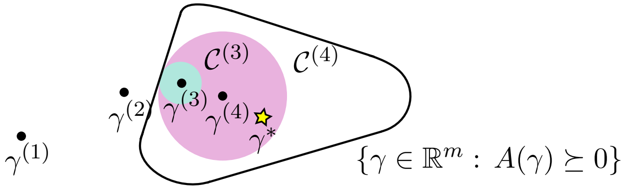

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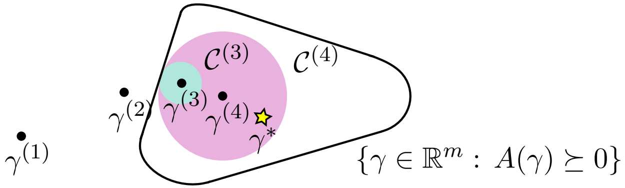

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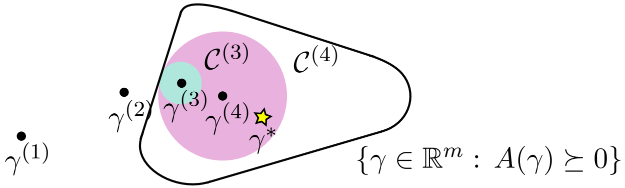

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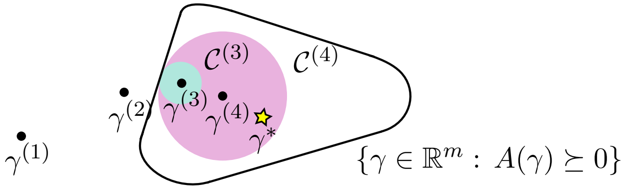

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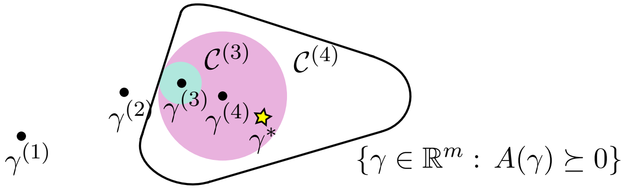

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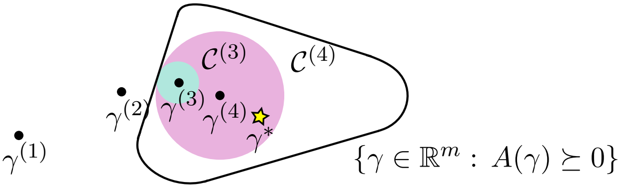
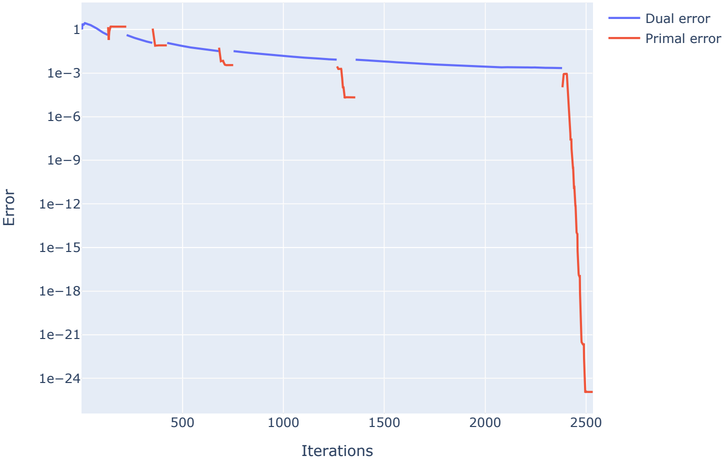


# CertSDP convergence behavior



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CertSDP produces iterates $X_t$ such that

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# Numerical results: experimental setup

- Random instances of **distance-minimization QMP**

$$\inf_{X\in\mathbb{R}^{n\times k}} \left\{\|X\|_F^2 : q_i(X) = 0,\ \forall i \in [m]\right\}$$

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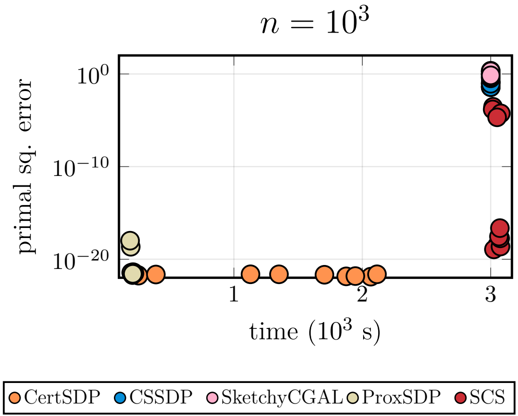
# Numerical results: convergence comparisons



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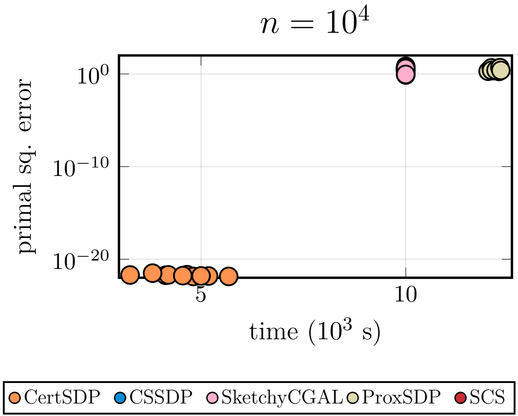

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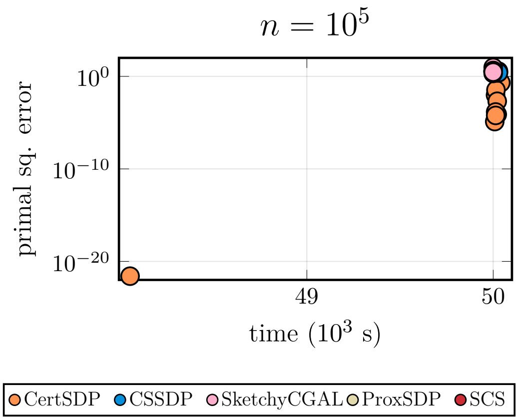
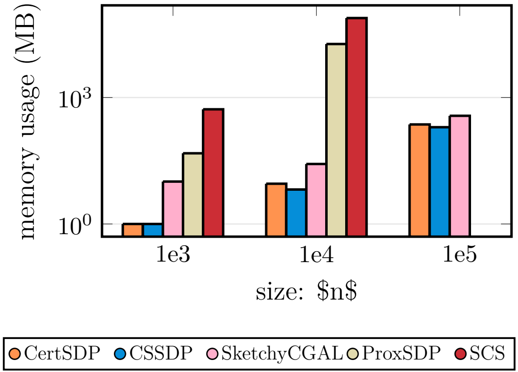


# Numerical results: memory usage



1. What structure? . . . $k$-exactness
2. How to use $k$-exactness? . . . Strongly convex reformulation
3. Algorithms
4. Numerical results
5. Conclusion

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**Thank you! Questions?**

`https://arxiv.org/abs/2206.00224`

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# The Burer–Monteiro method

$$(\textbf{SDP}) = \min_{Y \in \mathbb{S}^{n+k}} \left\{ \langle M_0, Y \rangle : \begin{array}{l} \langle M_i, Y \rangle + d_i = 0, \, \forall i \in [m] \\ Y \succeq 0 \end{array} \right\}$$

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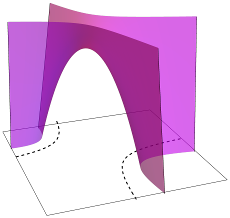

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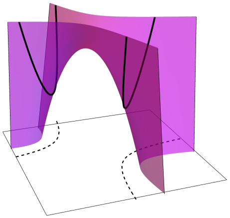

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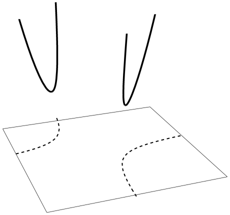

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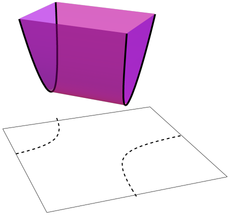

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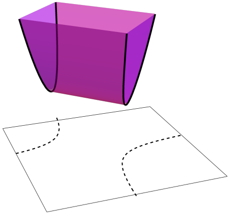

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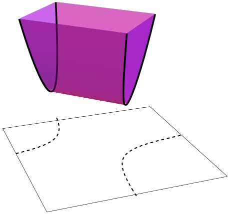

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- Too much symmetry! We set $\tilde{X} = \begin{pmatrix} X \\ I_k \end{pmatrix}$

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