A Convex Geometric Treatment of SDP Exactness in QCQPs

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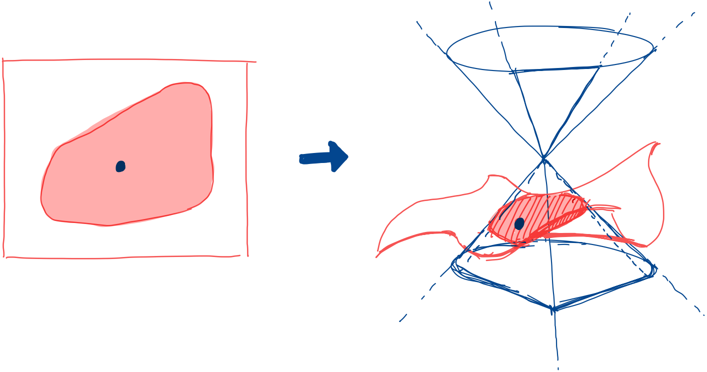

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1. Introduction
2. Geometric properties of $\overline{S}$
3. A general rounding procedure
4. Applications
5. Conclusion

Setup and goal

- Quadratically constrained set

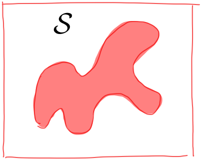
$$\mathcal{S} := \{x \in \mathbb{R}^n : \; q_i(x) = 0, \, \forall i \in [m]\}$$

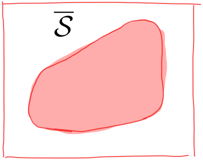
where $q_i(x) = x^\top A_i x + 2b_i^\top x + c_i$

- Assume exists $\gamma^* \in \mathbb{R}^m$ such that $\sum_{i=1}^m \gamma_i^* A_i \succ 0$
- Semidefinite relaxation

$$\overline{\mathcal{S}} := \left\{ x \in \mathbb{R}^n : \; \begin{array}{l} \exists\, X \succeq xx^\top \; : \\ \langle A_i, X \rangle + 2b_i^\top x + c_i = 0 \end{array} \right\}$$

- Goal: When does $\text{conv}(\mathcal{S}) = \overline{\mathcal{S}}$?





Outline

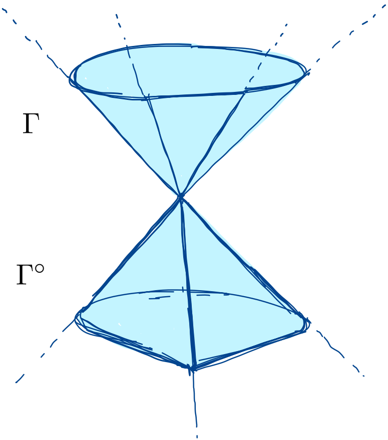
- Geometric properties of $\overline{\mathcal{S}}$
- General rounding procedure for: Given $x \in \overline{\mathcal{S}}$
  - Output convex decomposition for $x \in \operatorname{conv}(\mathcal{S})$
  - Or output $x' \in \overline{\mathcal{S}} \setminus \operatorname{conv}(\mathcal{S})$
- Applications
  - Quadratic matrix programming
  - Complementarity relations with binary variables

Related: [Yakubovich 71], [Laurent and Poljak 95], [Ramana and Goldman 95], [Ceria and Soares 99], [Pataki 00], [Beck 07], [Burer and Ye 19], [**W** and Kılınç-Karzan 20], [Eltved and Burer 20]

Rewriting the SDP

- *Convex Lagrange multipliers*

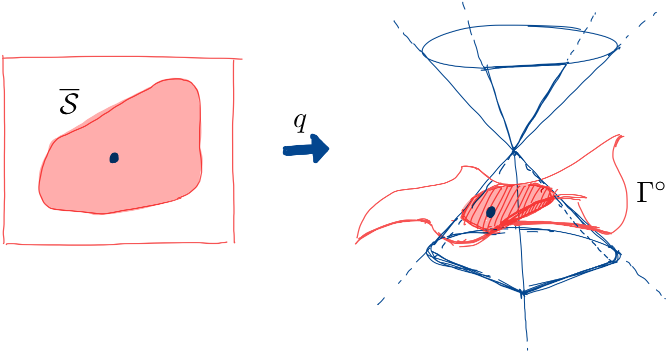
$$\Gamma := \left\{ \gamma \in \mathbb{R}^m : \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}.$$



Rewriting the SDP

Lemma ([W and Kılınç-Karzan 20])

$$\overline{\mathcal{S}} = \{x : \langle \gamma, q(x) \rangle \leq 0, \, \forall \gamma \in \Gamma\} = \{x : q(x) \in \Gamma^\circ\}.$$

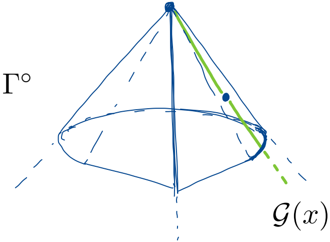


Useful geometric sets

- Let $x \in \overline{\mathcal{S}}$, i.e., $q(x) \in \Gamma^\circ$
- Define

$$\mathcal{G}(x) := \text{face}(\Gamma^\circ, q(x)),$$

i.e., $\mathcal{G}(x)$ is the *minimal face* of Γ° containing $q(x)$

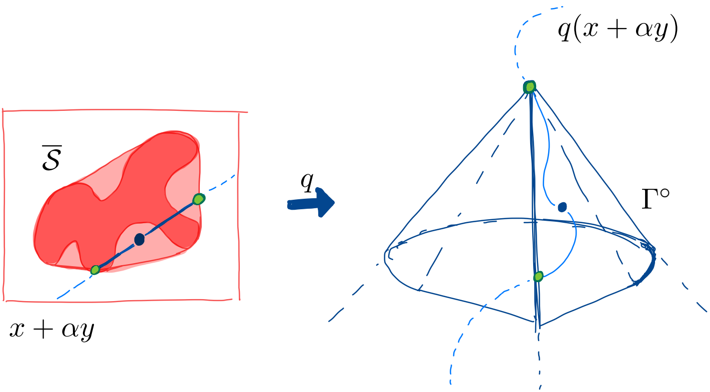


$\mathcal{G}(x)$

Useful geometric sets

- Define the *rounding directions*

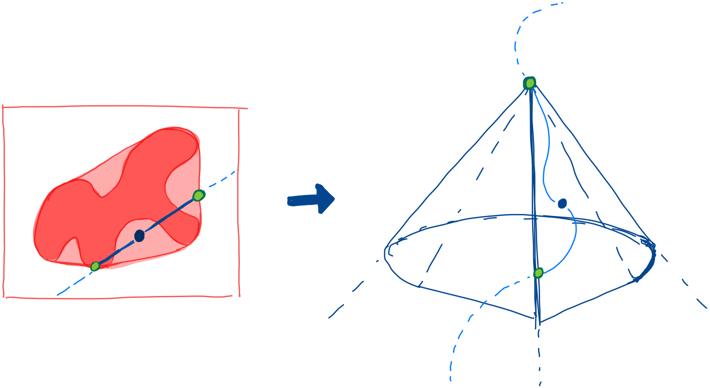
$$\mathcal{R}(x) \coloneqq \{y \in \mathbb{R}^n : \ q(x + \alpha y) \in \operatorname{span}(\mathcal{G}(x)), \ \forall \alpha \in \mathbb{R}\}$$



The rounding procedure

Procedure: Given $x \in \overline{\mathcal{S}}$

- If $x \in \mathcal{S}$, output x
- If $\mathcal{R}(x) = \{0\}$, output "Fail: $x \in \overline{\mathcal{S}} \setminus \operatorname{conv}(\mathcal{S})$"
- Else, let $y \in \mathcal{R}(x)$ nonzero and recurse on $x + \alpha_+ y$ and $x - \alpha_- y$

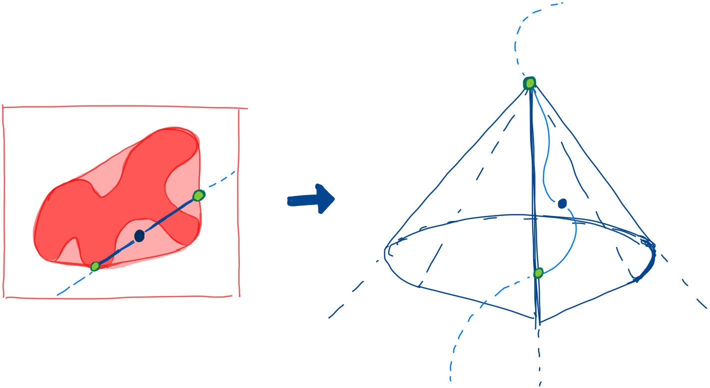


Some structural results

Lemma

$\dim(\mathcal{G}(x + \alpha_+ y)) < \dim(\mathcal{G}(x))$.

$\implies$ Procedure terminates after finitely many steps



Some structural results

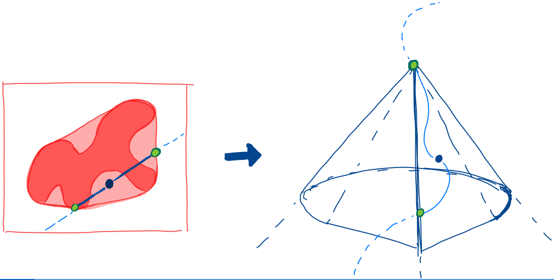
Lemma

If $x \in \mathrm{conv}(\mathcal{S}) \setminus \mathcal{S}$, then there exists y nonzero such that $[x \pm y] \subseteq \mathrm{conv}(\mathcal{S})$.

Lemma

Suppose Γ° is facially exposed. If $[x \pm y] \subseteq \overline{\mathcal{S}}$, then $y \in \mathcal{R}(x)$, the set of rounding directions.

$\implies \mathcal{R}(x) = \{0\}$ implies $x \notin \mathrm{conv}(\mathcal{S})$



Rounding procedure

Theorem

The rounding procedure terminates in finite time.

- Correctness: If rounding procedure outputs a convex decomposition, then $x \in \operatorname{conv}(\mathcal{S})$.
- Completeness: Suppose Γ° is facially exposed. If rounding procedure outputs “Fail” then $\operatorname{conv}(\mathcal{S}) \neq \overline{\mathcal{S}}$.

Quadratic matrix programming (QMP)

- Quadratic matrix sets

$$\left\{X \in \mathbb{R}^{n\times k}: \quad \operatorname{tr}(X^\top \mathbb{A}_i X) + 2\operatorname{tr}(X^\top B_i) + c_i = 0,\ \forall i \in [m] \right\}$$

$$\Longleftrightarrow$$

$$\left\{x \in \mathbb{R}^{nk}: \quad x^\top (I_k \otimes A_i) x + 2\operatorname{vec}(B_i)^\top x + c_i = 0,\ \forall i \in [m] \right\}$$

- Assuming this structure:

Lemma

Suppose $k \geq m+1$. Then for all $x \in \overline{\mathcal{S}} \setminus \mathcal{S}$, we have $\mathcal{R}(x) \neq \{0\}$.

- $\implies \overline{\mathcal{S}} = \operatorname{conv}(\mathcal{S})$ whenever $k \geq m+1$

Mixed binary programming

- Define

$$\mathcal{S} = \left\{ x \in \mathbb{R}^3 : \begin{array}{c} x_2^2 \leq x_3 \\ x_1 \in \{0,1\} \\ x_1 = 0 \implies x_2 = 0 \end{array} \right\}$$

Lemma

For all $x \in \overline{\mathcal{S}} \setminus \mathcal{S}$, we have $\mathcal{R}(x) \neq \{0\}$.

- $\implies \overline{\mathcal{S}} = \operatorname{conv}(\mathcal{S})$
- Bootstraps to multiple complementarity constraints $\longrightarrow$ sparse regression

Summary

- Quadratically constrained $\mathcal{S}$
- SDP relaxation $\overline{\mathcal{S}}$
- Geometry of $\overline{\mathcal{S}}$
  - $\mathcal{G}(x)$ the minimal face of Γ° containing $q(x)$
  - $\mathcal{R}(x)$ the set of *rounding directions*
- Procedure outputting $x \in \text{conv}(\mathcal{S})$ or $x' \in \overline{\mathcal{S}} \setminus \text{conv}(\mathcal{S})$
- Applications
  - Quadratic matrix programming
  - Complementarity relations with binary variables

Thank you. Questions?

Slides (and preprint hopefully soon) available online
`cs.cmu.edu/~alw1`

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