A linear-time algorithm for the generalized TRS based on a convex quadratic reformulation

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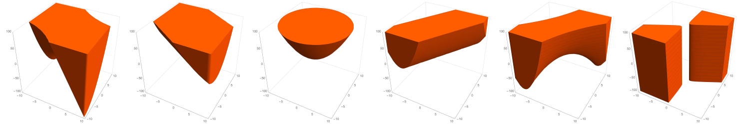


The Generalized Trust Region Subproblem (GTRS)

- $q_{\text{obj}}, q_{\text{cons}} : \mathbb{R}^n \to \mathbb{R}$ are **nonconvex** quadratic functions

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) \mid q_{\text{cons}}(x) \leq 0\}$$

- $q_{\text{obj}}(x) = x^\top A_{\text{obj}} x + 2b_{\text{obj}}^\top x + c_{\text{obj}}$
- $q_{\text{cons}}(x) = x^\top A_{\text{cons}} x + 2b_{\text{cons}}^\top x + c_{\text{cons}}$

Motivation

- Applications
  - Nonconvex quadratic integer programs, signal processing, compressed sensing, robust optimization, trust-region methods
- Surprisingly simple/beautiful theory
  - **Semidefinite programming** (SDP) relaxation is tight [FY79] $\implies$ polynomial-time algorithm
  - Connections between GTRS and **generalized eigenvalues**
  - Special instance of **quadratically-constrained quadratic programming** (QCQP)

Related work

- Convex **reformulations** of the GTRS in **lifted spaces**
  [BT96; BH14]
- **Algorithms** for the GTRS assuming **exact eigen-procedures**
  [PW14; FST18; JLW18; JL19; AN19]
- A **linear-time algorithm** for the GTRS
  [JL18]
- **Convex hull results**
  [Yıl09; MV17]

Results/Outline

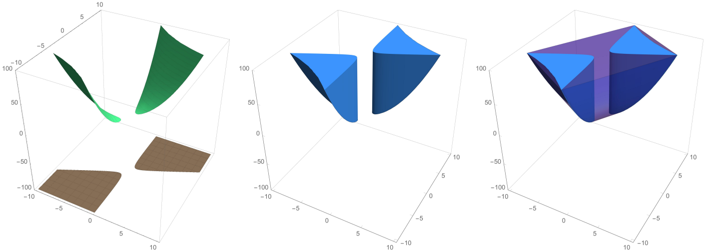
Under "mild assumptions"

- **Convex hull result** $\implies$ convex quadratic reformulation [JL19]
- New **linear-time algorithm** for approximating the GTRS
- Results extend to equality-, interval-, and hollow-constrained **variants** of the GTRS

1. Introduction
2. Convex hull result
3. Convex quadratic reformulation of the GTRS and algorithms

Suffices to optimize over convex hull of epigraph

$$
\begin{aligned}
&\inf_{x\in\mathbb{R}^n} \left\{ q_{\mathrm{obj}}(x) \,\middle|\, q_{\mathrm{cons}}(x) \le 0 \right\} \\
&\quad = \inf_{(x,t)\in\mathbb{R}^{n+1}} \left\{ t \,\middle|\, \begin{array}{l} q_{\mathrm{obj}}(x) \le t \\ q_{\mathrm{cons}}(x) \le 0 \end{array} \right\} =: \inf_{x,t} \left\{ t \,\middle|\, (x,t) \in \mathcal{S} \right\} \\
&\quad = \inf_{x,t} \left\{ t \,\middle|\, (x,t) \in \overline{\mathrm{conv}}(\mathcal{S}) \right\}
\end{aligned}
$$



A pencil of quadratics

- Main object of analysis

$$q(\gamma, x) \coloneqq q_{\text{obj}}(x) + \gamma q_{\text{cons}}(x)$$

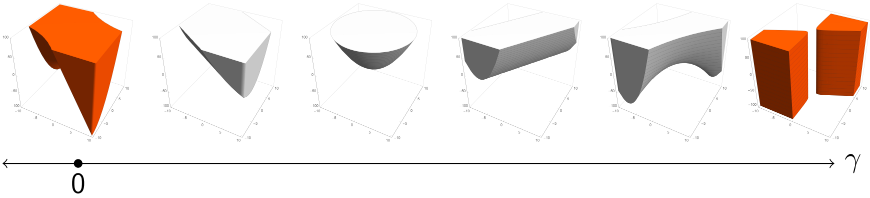
- $A(\gamma) \coloneqq A_{\text{obj}} + \gamma A_{\text{cons}}$
- $\mathcal{S}(\gamma) \coloneqq \{(x, t) \mid q(\gamma, x) \leq t\}$

Exercise

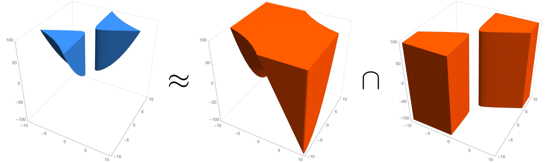
$\mathcal{S} = \bigcap_{\gamma \geq 0} \mathcal{S}(\gamma)$

$$\left\{(x, t) \,\middle|\, \begin{array}{l} q_{\text{obj}}(x) \leq t \\ q_{\text{cons}}(x) \leq 0 \end{array} \right\} \overset{?}{=} \subseteq \supseteq = \bigcap_{\gamma \geq 0} \{(x, t) \mid q_{\text{obj}}(x) + \gamma q_{\text{cons}}(x) \leq t\}$$

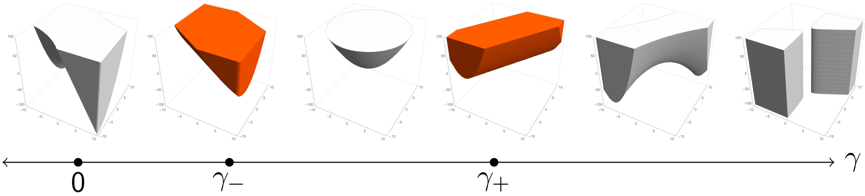
A pencil of quadratics



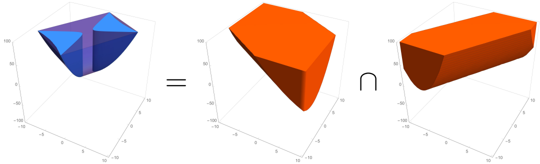
- $\mathcal{S}(\gamma) \coloneqq \{(x,t) \,|\, q(\gamma,x) \le t\}$
  - $\mathcal{S}(0) = \{(x,t) \,|\, q_{\text{obj}}(x) \le t\}$
  - $\mathcal{S}(\text{large number}) \approx \{(x,t) \,|\, q_{\text{cons}}(x) \le 0\}$
  - $\mathcal{S} \approx \mathcal{S}(0) \cap \mathcal{S}(\text{large number})$



Convex hull result



- Define $\Gamma := \{\gamma \geq 0 \,|\, \mathcal{S}(\gamma) \text{ is convex}\} = \{\gamma \geq 0 \,|\, A(\gamma) \succeq 0\}$
- Define $[\gamma_-, \gamma_+] := \Gamma$
- $\overline{\text{conv}}(\mathcal{S}) = \mathcal{S}(\gamma_-) \cap \mathcal{S}(\gamma_+)$

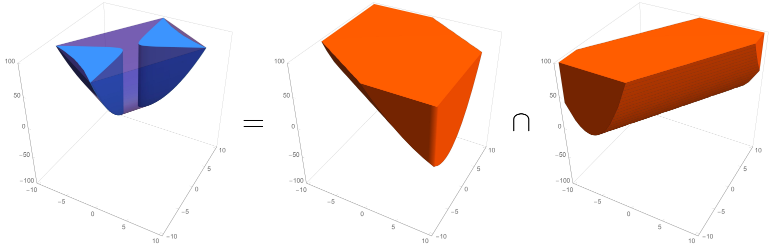


Convex hull result

Theorem

Suppose q_{obj} and q_{cons} are both nonconvex and Γ is nonempty, i.e., there exists $\gamma \geq 0$ such that $A(\gamma) \succeq 0$. Then Γ can be written $\Gamma = [\gamma_-, \gamma_+]$ and

$$\overline{\text{conv}}(\mathcal{S}) = \mathcal{S}(\gamma_-) \cap \mathcal{S}(\gamma_+) = \left\{ (x,t) \;\middle|\; \begin{array}{l} q(\gamma_-, x) \leq t \\ q(\gamma_+, x) \leq t \end{array} \right\}$$



Convex quadratic reformulation

- We can reformulate

$$\begin{aligned}
\mathrm{Opt} &= \inf_{(x,t)} \left\{ t \,\middle|\, (x,t) \in \overline{\mathrm{conv}}(\mathcal{S}) \right\} \\
&= \inf_{(x,t)} \left\{ t \,\middle|\, \begin{array}{l} q(\gamma_-, x) \leq t \\ q(\gamma_+, x) \leq t \end{array} \right\} \\
&= \inf_{x} \max \left\{ q(\gamma_-, x), q(\gamma_+, x) \right\}
\end{aligned}$$

- Algorithmic challenges
  - γ_- and γ_+ are not given
  - Only have **approximate (generalized) eigen-procedures**
  - Smooth minimax framework [Nes18] requires efficiently computing a **prox mapping**
  - How to recover a solution to the GTRS?

A linear-time algorithm for the GTRS

- Algorithm idea (assume A_0 and A_1 are "well-conditioned")
  - N is the number of nonzero entries in A_0 and A_1
  - p is the failure probability
  - Approximate γ_- and γ_+ to "high enough accuracy"

$$\tilde{O}\left(\frac{N}{\sqrt{\epsilon}}\log\left(\frac{n}{p}\right)\log\left(\frac{1}{\epsilon}\right)\right)$$

  - Approximately solve a smooth minimax problem

$$O\left(\frac{N}{\sqrt{\epsilon}}\right)$$

A linear-time algorithm for the GTRS

Theorem

There exists an algorithm, which given nonconvex quadratics q_{obj} and q_{cons} satisfying

- *there exists $\gamma \geq 0$ such that $A(\gamma) \succ 0$ and*
- *"mild" regularity assumptions,*

outputs an ϵ-approximate optimizer to the GTRS with probability $\geq 1 - p$. This algorithm runs in time

$$\approx \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right)$$

where N is the number of nonzero entries in A_{obj} and A_{cons}.

Recap

- Want to optimize the GTRS: $\inf_x \{q_{\text{obj}}(x) \mid q_{\text{cons}}(x) \leq 0\}$
- Studied a pencil of quadratics $q(\gamma, x)$
- Gave an explicit description of $\overline{\text{conv}}(\mathcal{S})$
- Convex quadratic reformulation!
- Gave a linear (in N) time algorithm for solving the GTRS and its variants

Future directions

- Can we generalize techniques in this paper to handle more than one quadratic constraints?
- What are the "right" regularity parameters?

Thank you. Questions?

Slides and Mathematica notebook
cs.cmu.edu/~alw1/iccopt.html

Preprint

Alex L. Wang and Fatma Kılınç-Karzan. *The Generalized Trust Region Subproblem: solution complexity and convex hull results*. Tech. rep. arXiv:1907.08843. ArXiV, 2019. URL: arxiv.org/abs/1907.08843

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