Hardy-Muckenhoupt Bounds for Laplacian Eigenvalues

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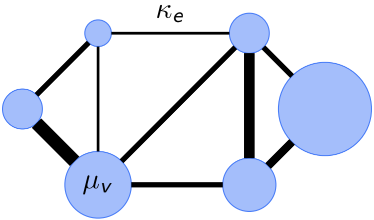
Based on joint work with Gary Miller, Noel Walkington

Introduction: graphs and the Laplacian

- Weighted connected graph, $G = (V, E, \mu, \kappa)$
- Mass, $\mu \in \mathbb{R}^V_{>0}$
- Spring constants, $\kappa \in \mathbb{R}^E_{>0}$



- Degrees, $d_v = \sum_{u \sim v} \kappa_{u,v}$

Introduction: graphs and the Laplacian

- Laplacian, $L = D - A$
- Degree matrix, $D = \mathrm{diag}(d_v)$
- Adjacency matrix, $A_{u,v} = \kappa_{u,v}$

Introduction: graphs and the Laplacian

- Let $x \in \mathbb{R}^V$
- As a linear map

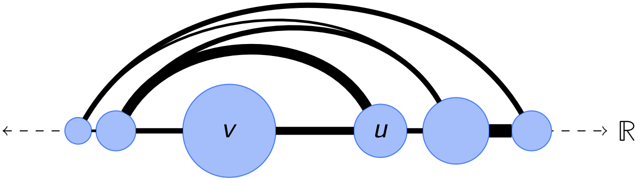
$$
\begin{aligned}
(Lx)_v &= (Dx)_v - (Ax)_v \\
&= \left(\sum_{u \sim v} \kappa_{u,v}\right) x_v - \sum_{u \sim v} \kappa_{u,v} x_u \\
&= \sum_{u \sim v} \kappa_{u,v}(x_v - x_u)
\end{aligned}
$$

- As a quadratic form

$$
\begin{aligned}
x^\top Lx &= \dots \\
&= \sum_{(u,v) \in E} \kappa_{u,v}(x_u - x_v)^2
\end{aligned}
$$

Introduction: spring mass systems

- Let's embed G on $\mathbb{R}$ as $v \mapsto x_v \in \mathbb{R}$



- Force acting on v is $\sum_{u \sim v} \kappa_{(u,v)}(x_u - x_v) = -(Lx)_v$
- Standing wave equation

$$
\begin{aligned}
-\text{acceleration}_v &= \lambda x_v, \qquad \forall v \in V \\
\frac{(Lx)_v}{\mu_v} &= \lambda x_v, \qquad \forall v \in V \\
(Lx)_v &= \lambda \mu_v x_v, \qquad \forall v \in V \\
Lx &= \lambda M x
\end{aligned}
$$

- $\sqrt{\lambda} \propto$ frequency

Introduction: spring mass systems

- $Lx = \lambda Mx$ has eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_{|V|}$
- 1 is eigenvector with eigenvalue 0
- Neumann problem is to find λ_2
- Courant-Fischer,

$$\implies \lambda_2 = \min_x \left\{ \frac{x^\top Lx}{x^\top Mx} \,\middle|\, x^\top M1 = 0 \right\}$$

- Rayleigh quotient,

$$\frac{x^\top Lx}{x^\top Mx} = \frac{\sum_{(u,v)\in E} \kappa_{u,v}(x_u - x_v)^2}{\sum_{v\in V} \mu_v x_v^2}$$

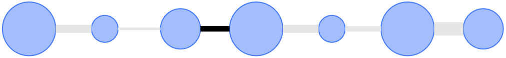
Introduction: λ_2

- Mixing time of random walks
- Markov chains
- Laplacian solvers
- Image segmentation
- Uncertainty principles
- Heat flow

Introduction: cuts and Cheeger's inequality

- Sparsest cut of G,

$$\text{S-CUT}(G) = \min_A \left\{ \frac{\sum_{e \in E(A,\bar{A})} \kappa_e}{\min(\mu(A), \mu(\bar{A}))} \,\middle|\, A, \bar{A} \neq \varnothing \right\}$$



- Numerator small $\approx A$ and $\bar{A}$ are sparsely connected
- Denominator large $\approx A$ and $\bar{A}$ both have a lot of mass

Introduction: cuts and Cheeger's inequality

- Cheeger's inequality:

Theorem

If $\mu_v = d_v$, then

$$\frac{\lambda_2}{2} \leq S\text{-}CUT \leq \sqrt{2\lambda_2}$$

or equivalently

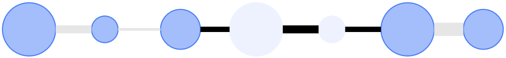
$$\frac{S\text{-}CUT^2}{2} \leq \lambda_2 \leq 2S\text{-}CUT$$

- Both sides are tight (up to constants) ☹

Introduction: our work

- Neumann content, Ψ_2

$$\Psi_2 \approx \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\min(\mu(A),\mu(B))} \,\middle|\, A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$



- $\kappa_{\text{eff}}(A, \bar{A}) = \sum_{e \in E(A,\bar{A})} \kappa_e$
- S-CUT must partition, Ψ_2 may leave out vertices

Introduction: our work

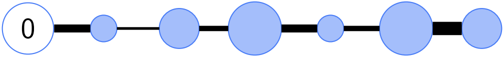
Theorem (Main theorem)

Let G be a weighted connected graph. Then,

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2$$

Dirchlet problem: the path graph

- Vertices, $\{v_0, v_1, v_2, \ldots, v_n\}$
- Edges, $\{(v_{i-1}, v_i) \mid i \in [n]\}$
- Mass of vertex v_i is $\mu_i > 0$
- Spring constant of edge (v_{i-1}, v_i) is $\kappa_i > 0$
- Want to solve standing wave equation where v_0 is held at 0



Dirchlet problem: the path graph

- Dirichlet problem is to find λ

$$\begin{aligned}\lambda &= \min_x \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x_0 = 0 \right\} \\ &= \min_x \left\{ \frac{\sum_{i=1}^n \kappa_i (x_i - x_{i-1})^2}{\sum_{i=1}^n \mu_i x_i^2} \,\middle|\, x_0 = 0 \right\}\end{aligned}$$

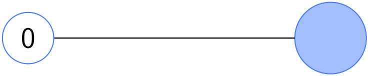
Dirchlet problem: for two node graphs

- Suppose $n = 1$,

$$\begin{aligned}\lambda &= \min_{x_0, x_1} \left\{ \frac{\kappa_1 (x_1 - x_0)^2}{\mu_0 x_0^2 + \mu_1 x_1^2} \,\middle|\, x_0 = 0 \right\} \\ &= \min_{x_1} \left\{ \frac{\kappa_1 x_1^2}{\mu_1 x_1^2} \right\} \\ &= \frac{\kappa_1}{\mu_1}\end{aligned}$$

D. problem: effective spring constants and the D. content

- Pick $k \in [n]$, let $A_k = \{v_k, v_{k+1}, \ldots, v_n\}$
- Consider graph G_k



  - $u_0 \sim u_{A_k}$
  - $\mu(u_{A_k}) = \mu(A_k)$
  - $\kappa(u_0, u_{A_k}) = \kappa_{\text{eff}}(v_0, v_k) = \left(\sum_{i=1}^{k} \kappa_i^{-1}\right)^{-1}$
- $\lambda(G_k) = \frac{\kappa_{\text{eff}}(v_0, v_k)}{\mu(A_k)}$

D. problem: effective spring constants and the D. content

- Define Dirichlet content

$$\Psi = \min_k \lambda(G_k) = \min_k \frac{\kappa_{\text{eff}}(v_0, v_k)}{\mu(A_k)}$$

D. problem: Muckenhoupt's inequality

Corollary (Muckenhoupt, 1972)

Let $\mu, \kappa \in \mathbb{R}^n_{>0}$. *Let* C *be the smallest constant such that for all* $y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{i} y_j \right)^2 \leq C \sum_{i=1}^{n} \kappa_i y_i^2.$$

Let

$$B = \max_{1 \leq k \leq n} \left(\sum_{i=1}^{k} \kappa_i^{-1} \right) \left(\sum_{i=k}^{n} \mu_i \right).$$

Then $B \leq C \leq 4B$.

D. problem: Muckenhoupt's inequality

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Then $B \leq C \leq 4B$.

Suppose $x \in \mathbb{R}^V$ with $x_0 = 0$ and define $y_i = x_i - x_{i-1}$.

D. problem: Muckenhoupt's inequality

Corollary

Let $\mu, \kappa \in \mathbb{R}^n_{>0}$. *Let* C *be the smallest constant such that for all* $x \in \mathbb{R}^V$ *with* $x_0 = 0$,

$$\frac{1}{C} \leq \frac{\sum_{i=1}^n \kappa_i (x_i - x_{i-1})^2}{\sum_{i=1}^n \mu_i x_i^2}.$$

Let

$$B = \max_{1 \leq k \leq n} \left(\sum_{i=1}^k \kappa_i^{-1} \right) \left(\sum_{i=k}^n \mu_i \right).$$

Then $B \leq C \leq 4B$.

D. problem: Muckenhoupt's inequality

Corollary

Let $\mu, \kappa \in \mathbb{R}^n_{>0}$. *Let* C *be*

$$\frac{1}{C} = \min_{x \in \mathbb{R}^V} \left\{ \frac{\sum_{i=1}^n \kappa_i (x_i - x_{i-1})^2}{\sum_{i=1}^n \mu_i x_i^2} \,\middle|\, x_0 = 0 \right\}.$$

Let

$$\frac{1}{B} = \min_{1 \le k \le n} \frac{\left(\sum_{i=1}^k \kappa_i^{-1}\right)^{-1}}{\sum_{i=k}^n \mu_i}.$$

Then $\frac{1}{4}\frac{1}{B} \le \frac{1}{C} \le \frac{1}{B}$.

$\frac{1}{C} = \lambda$ and $\frac{1}{B} = \Psi$

D. problem: Muckenhoupt's inequality

Corollary

Let G be a weighted connected path graph. Let λ be the Dirichlet eigenvalue and let Ψ be the Dirichlet content of G. Then,

$$\frac{\Psi}{4} \leq \lambda \leq \Psi.$$

D. problem: Muckenhoupt's inequality

(Very sketchy) proof sketch.

$$\sum_i \mu_i x_i^2$$

□

Goal: $\leq \frac{4}{\Psi} \sum_i \kappa_i (x_i - x_{i-1})^2$.

D. problem: Muckenhoupt's inequality

(Very sketchy) proof sketch.

$$\sum_i \mu_i x_i^2 = \sum_i \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1}) \right)^2$$

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$$\leq \sum_i \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1})^2 \right) \left(\sum_{j=1}^{i} 1^2 \right)$$

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$$\sum_i \mu_i x_i^2 = \sum_i \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1}) \frac{\sqrt{\kappa_j}}{\sqrt{\kappa_j}} \right)^2$$

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$$\leq \sum_i \mu_i \left(\sum_{j=1}^{i} \kappa_j (x_j - x_{j-1})^2 \right) \left(\sum_{j=1}^{i} \frac{1}{\kappa_j} \right)$$

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☹

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D. problem: Muckenhoupt's inequality

(Very sketchy) proof sketch.

$$\sum_i \mu_i x_i^2$$

$\square$

Goal: $\leq \frac{4}{\Psi} \sum_i \kappa_i (x_i - x_{i-1})^2$.

D. problem: Muckenhoupt's inequality

(Very sketchy) proof sketch.

$$\sum_i \mu_i x_i^2 = \sum_i \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1}) \frac{\sqrt{\alpha_j \kappa_j}}{\sqrt{\alpha_j \kappa_j}} \right)^2$$

□

Goal: $\leq \frac{4}{\Psi} \sum_i \kappa_i (x_i - x_{i-1})^2$.

D. problem: Muckenhoupt's inequality

(Very sketchy) proof sketch.

$$\sum_i \mu_i x_i^2 = \sum_i \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1}) \frac{\sqrt{\alpha_j \kappa_j}}{\sqrt{\alpha_j \kappa_j}} \right)^2$$

$$\leq \sum_i \mu_i \left(\sum_{j=1}^{i} \alpha_j \kappa_j (x_j - x_{j-1})^2 \right) \left(\sum_{j=1}^{i} \frac{1}{\alpha_j \kappa_j} \right)$$

□

Goal: $\leq \frac{4}{\Psi} \sum_i \kappa_i (x_i - x_{i-1})^2$.

D. problem: Muckenhoupt's inequality

(Very sketchy) proof sketch.

$$\sum_i \mu_i x_i^2 = \sum_i \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1}) \frac{\sqrt{\alpha_j \kappa_j}}{\sqrt{\alpha_j \kappa_j}} \right)^2$$

$$\leq \sum_i \mu_i \left(\sum_{j=1}^{i} \alpha_j \kappa_j (x_j - x_{j-1})^2 \right) \left(\sum_{j=1}^{i} \frac{1}{\alpha_j \kappa_j} \right)$$

Pick α_j so that Cauchy-Schwarz is tight when $x_i = \sqrt{\sum_{j=1}^{i} \frac{1}{\kappa_i}}$ □

Goal: $\leq \frac{4}{\Psi} \sum_i \kappa_i (x_i - x_{i-1})^2$.

D. problem: Muckenhoupt's inequality

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☺

Pick α_j so that Cauchy-Schwarz is tight when $x_i = \sqrt{\sum_{j=1}^{i} \frac{1}{\kappa_i}}$ □

Goal: $\leq \frac{4}{\Psi} \sum_i \kappa_i (x_i - x_{i-1})^2$.

Neumann problem: the path graph

- Neumann problem is to find

$$\begin{aligned}\lambda_2 &= \min_x \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x^\top M 1 = 0 \right\} \\ &= \min_x \left\{ \frac{\sum_{i=2}^n \kappa_{(i,i-1)}(x_i - x_{i-1})^2}{\sum_{i=1}^n \mu_i x_i^2} \,\middle|\, \sum \mu_i x_i = 0 \right\}\end{aligned}$$

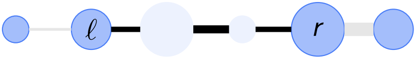
Neumann problem: the path graph

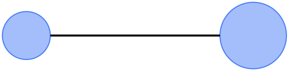
- Suppose $n = 2$. Then,

$$\begin{aligned}\lambda_2 &= \min_{x_1, x_2} \left\{ \frac{\kappa_{1,2}(x_1 - x_2)^2}{\mu_1 x_1^2 + \mu_2 x_2^2} \,\middle|\, \mu_1 x_1 + \mu_2 x_2 = 0 \right\} \\ &= \dots \\ &= \frac{\kappa_{1,2}}{(\mu_1^{-1} + \mu_2^{-1})^{-1}} \\ &\approx \frac{\kappa_{1,2}}{\min(\mu_1, \mu_2)}\end{aligned}$$

Neumann problem: the N. content

- Pick $1 \le \ell < r \le n$
- Consider graph $G_{\ell,r}$





  - $u_\ell \sim u_r$
  - $\mu(u_\ell) = \sum_{i=1}^{\ell} \mu_i$
  - $\mu(u_r) = \sum_{i=r}^{n} \mu_i$
  - $\kappa(u_\ell, u_r) = \kappa_{\text{eff}}(v_\ell, v_r)$
- $\lambda_2(G_{\ell,r}) \approx \frac{\kappa_{\text{eff}}(v_l, v_r)}{\min(\mu_\ell, \mu_r)}$

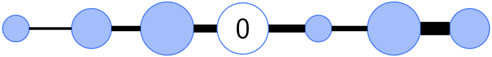
Neumann problem: the path graph and the N. content

- Define N. content

$$\Psi_2 = \min_{1 \le \ell < r \le n} \lambda_2(G_{\ell,r})$$
$$\approx \min_{1 \le \ell < r \le n} \frac{\kappa_{\text{eff}}(u_\ell, u_r)}{\min(\mu(u_\ell), \mu(u_r))}$$

Neumann problem: as two Dirichlet problems

- Pick pinch point $p \in (1, n)$
- Then, split into two path graphs G_-, G_+



Lemma

$$\lambda_2(G) = \min_{p \in (1,n)} \max\left(\lambda(G_-), \lambda(G_+)\right)$$

(Hint: use Courant-Fischer)

Neumann problem: proof of main theorem

Theorem

$\Psi_2/4 \leq \lambda_2 \leq \Psi_2$.

Proof sketch.

$$\begin{aligned}
\lambda_2 &= \min_{p\in(1,n)} \max\left(\lambda(G_-), \lambda(G_+)\right) \\
&\asymp \min_{p\in(1,n)} \min_{1\leq\ell<p<r\leq n} \max\left(\frac{\kappa_{\text{eff}}(v_\ell, p)}{\mu(u_\ell)}, \frac{\kappa_{\text{eff}}(p, v_r)}{\mu(u_r)}\right) \\
&= \min_{1\leq\ell<r\leq n} \min_{\ell<p<r} \max\left(\frac{\kappa_{\text{eff}}(v_\ell, p)}{\mu(u_\ell)}, \frac{\kappa_{\text{eff}}(p, v_r)}{\mu(u_r)}\right) \\
&= \min_{1\leq\ell<r\leq n} \frac{\kappa_{\text{eff}}(v_\ell, v_r)}{(\mu(u_\ell)^{-1} + \mu(u_r)^{-1})^{-1}} \\
&= \Psi_2.
\end{aligned}$$

□

Generalizations: arbitrary connected graphs

Theorem (Dirichlet on a graph)

Let G be a weighted connected graph. Let $S \subseteq V$ be a proper nonempty set. Let $\lambda(G, S)$ be the Dirichlet eigenvalue and let $\Psi(G, S)$ be the Dirichlet content of G. Then

$$\frac{\Psi}{4} \leq \lambda \leq \Psi.$$

Theorem (Neumann on a graph)

Let G be a weighted connected graph. Let $\lambda_2(G)$ be the Neumann eigenvalue and let $\Psi_2(G)$ be the Neumann content of G. Then

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2.$$

Generalizations: p-Laplacian

Theorem

Let G be a weighted connected graph. Let $\lambda_2(G)$ be the Neumann eigenvalue of the p-Laplacian and let $\Psi_2(G)$ be the p-Neumann content of G. Then

$$\frac{\Psi_2}{pq^{p/q}} \leq \lambda_2 \leq \Psi_2.$$

Summary

- Muckenhoupt's weighted Hardy inequality
- Neumann content,

$$\Psi_2 \approx \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\min(\mu(A), \mu(B))} \,\middle|\, A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$

- Showed

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2$$

- Approximation algorithms?