# Accurately and efficiently solving structured nonconvex optimization problems

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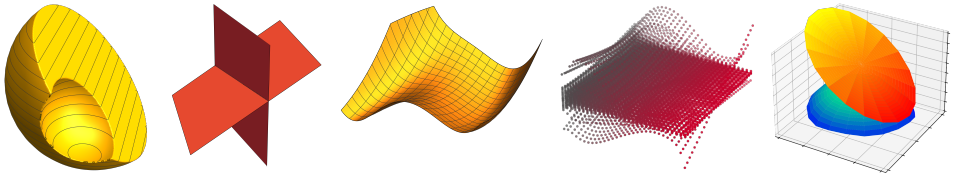


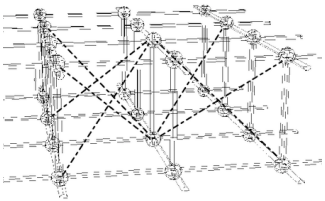
These slides are publicly available at `cs.cmu.edu/~alw1`

# Collaborators

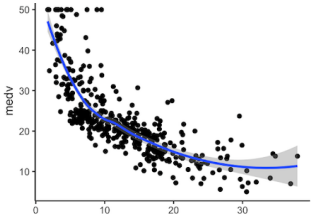
Carnegie Mellon University (OR, Math, CS),
Northwestern University, Fudan University,
Peking University

# Convex optimization

- Convex optimization is influential in many different fields



**Engineering**
Controller stability, power allocation, truss design, +




**Statistics**
(Linear) Regression, parameter estimation, +

**Finance**
Portfolio optimization, risk analysis, +

- Convex optimization is accurate and efficient

# Convex optimization, meet nonconvex problems

- Unfortunately, many practical optimization problems are nonconvex
- Example: Low-rank matrix completion (**Netflix problem**)

Movies

Users

$$\begin{pmatrix} 5 & 4 & ? & ? & 4 \\ 3 & ? & ? & 3 & ? \\ ? & 2 & 4 & 1 & 1 \\ ? & 3 & ? & ? & 4 \end{pmatrix}$$

$$\min_{X \in \mathbb{R}^{n \times k}} \{\mathrm{rank}(X) : X \text{ agrees with revealed entries}\}$$

- Rank constraints, binary constraints, sparsity constraints
- Generally hard, **but not always!**
- Some nonconvex problems can be solved using convex optimization

# Research goal

## Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Completed work**:
  - Nonconvex problems: quadratically constrained quadratic programs (QCQPs)
  - Convex relaxations: semidefinite programs (SDPs)

## Today's questions

Understand **structures within QCQPs** that enable us to solve them exactly and efficiently **using SDPs**

# Today's outline

- **Preliminaries**
  QCQPs and their applications, the SDP relaxation
- **Understand structures within QCQPs that enable us to solve them**...
  - **exactly** [IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*]
    Objective value, convex hull exactness, applications
  - **efficiently** [Math. Prog. 20], [SIAM J. Optim. *under review*], [Ongoing]
    The generalized trust-region subproblem and regular QCQPs
- **Conclusion and future directions**

1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

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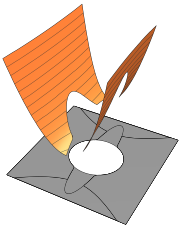
# Quadratically constrained quadratic programs (QCQPs)

- $q_{\text{obj}}, q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ quadratic (possibly nonconvex!)

$$q_i(x) = x^\mathsf{T} A_i x + 2b_i^\mathsf{T} x + c_i$$

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \left\{ q_{\text{obj}}(x) : \; q_i(x) \leq 0, \, \forall i \in [m] \right\}$$

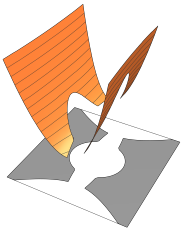


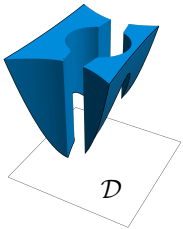
- Highly expressive:
  - MAX-CUT, MAX-CLIQUE, pooling, truss design, facility location, production planning
  - binary programs $x_1(1 - x_1) = 0$
  - polynomial optimization problems $x_1 x_2 = z_{12}$
- NP-hard in general

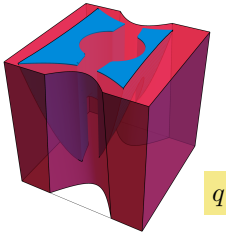
# The QCQP epigraph

- QCQP epigraph

$$\mathcal{D} := \left\{ (x,t) \in \mathbb{R}^{n+1} : \begin{array}{l} q_{\text{obj}}(x) \le t \\ q_i(x) \le 0,\ \forall i \in [m] \end{array} \right\}$$





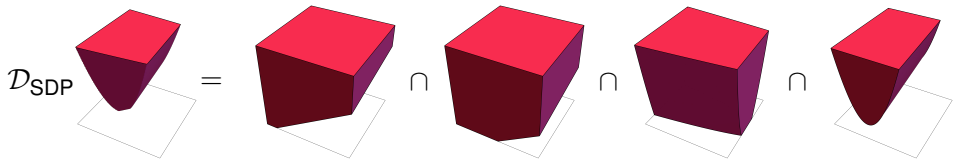




- How can we derive ~~convex~~ relaxations of $\mathcal{D}$?
- If $\gamma \in \mathbb{R}^m_+$, then

$$\forall (x,t) \in \mathcal{D}, \quad \underbrace{q_{\text{obj}}(x) + \sum_{i=1}^{m} \gamma_i q_i(x)}_{=: q(\gamma, x)} \le t$$

# The SDP relaxation

- SDP relaxation = impose all convex aggregated inequalities!




- Formally,

$$\Gamma := \{\gamma \in \mathbb{R}^m_+ : \ q(\gamma, x) \text{ is convex in } x\} = \left\{\gamma \in \mathbb{R}^m_+ : \ A_{\text{obj}} + \sum_{i=1}^m \gamma_i A_i \succeq 0\right\}$$

$$\mathcal{D}_{\text{SDP}} := \bigcap_{\gamma \in \Gamma} \{(x,t) : \ q(\gamma, x) \le t\} = \left\{(x,t) \in \mathbb{R}^{n+1} : \ \sup_{\gamma \in \Gamma} q(\gamma, x) \le t\right\}$$

$$\text{Opt}_{\text{SDP}} := \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x)$$

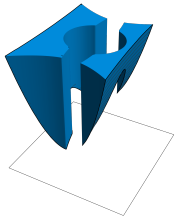
# The usual SDP relaxation

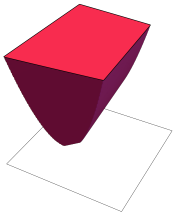
$$
\begin{aligned}
&\inf_{x\in\mathbb{R}^n} \{q_{\mathsf{obj}}(x) : q_i(x) \le 0,\ \forall i \in [m]\} \\
&\quad = \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\mathsf{obj}}, X\rangle + 2b_{\mathsf{obj}}^\top x + c_{\mathsf{obj}} : \begin{array}{l} X = xx^\top \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0,\ \forall i \in [m] \end{array} \right\} \\
&\quad \ge \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\mathsf{obj}}, X\rangle + 2b_{\mathsf{obj}}^\top x + c_{\mathsf{obj}} : \begin{array}{l} X - xx^\top \succeq 0 \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0,\ \forall i \in [m] \end{array} \right\} \\
&\quad = \inf_{x\in\mathbb{R}^n} \inf_{X\in\mathbb{S}^n} \dots \\
&\quad = \inf_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q(\gamma, x)
\end{aligned}
$$

# Preliminaries recap

- Main objects of interest

| | nonconvex QCQP | convex SDP |
|---|---|---|
| Optimum value | $\mathrm{Opt}$ | $\mathrm{Opt}_{\mathrm{SDP}}$ |
| Epigraph | $\mathcal{D}$ | $\mathcal{D}_{\mathrm{SDP}}$ |





- Useful for analysis:
  - $q(\gamma, x) =$ Lagrangian function
  - $\Gamma =$ aggregation weights giving convex $q(\gamma, x)$

1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

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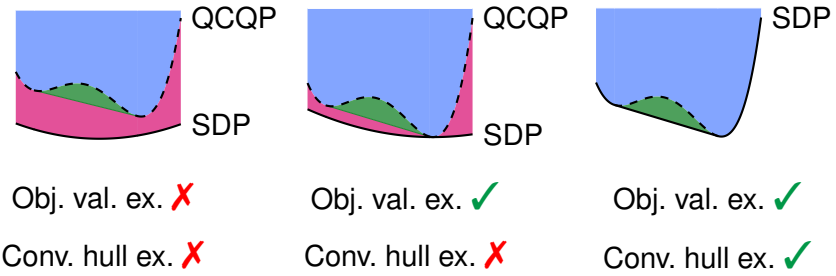
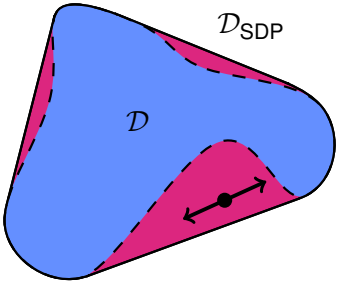
# Forms of exactness

- What does exactness mean?
  - Objective value exactness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$
  - Convex hull exactness: $\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}}$ $\leftarrow$ convexification of substructures



# Convex hull exactness

- $\mathrm{conv}(\mathcal{D}) \stackrel{?}{=} \mathcal{D}_{\mathsf{SDP}}$




$\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}} \iff$ "Given any point in $\mathcal{D}_{\mathsf{SDP}} \setminus \mathcal{D}$, exists direction such that can move forward and backward inside $\mathcal{D}_{\mathsf{SDP}}$"

- Can carry out this idea for QCQPs!
- Sufficient conditions based on abstract properties $\longrightarrow$ concrete conditions

# Sufficient conditions based on abstract properties

## Theorem ([Math. Prog. 21])

Suppose $\Gamma$ polyhedral. If for every semidefinite face $\mathcal{F} \trianglelefteq \Gamma$,

$$\operatorname{aff}\left(\operatorname{Proj}_{\mathcal{V}(\mathcal{F})}\{b(\gamma) : \gamma \in \mathcal{F}\}\right) \neq \mathcal{V}(\mathcal{F}),$$

then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$.

## Theorem ([Math. Prog. *under review*])

If for every $(x,t) \in \mathcal{D}_{\text{SDP}} \setminus \mathcal{D}$,

$$\left\{ (x',t') \in \mathbb{R}^{n+1} : \begin{array}{l} x' \in \mathcal{V}(\mathcal{F}) \\ \langle A(\eta)x + b(\eta), x' \rangle - t' = 0,\ \forall (1,\eta) \in (\mathcal{F}^{\triangle})^{\perp} \end{array} \right\} \neq \{0\},$$

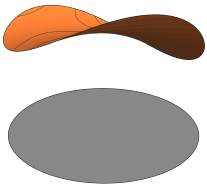
then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$.

Based on: **[IPCO 19]**, **[Math. Prog. 21]**, **[Math. Prog. *under review*]**

# Example: the trust-region subproblem

- Convex hull exactness in the case of single ball constraint

$$\text{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ q_{\text{obj}}(x) : \|x\|^2 \leq 1 \right\}$$



- **Applications**:
  - Nonlinear minimization (trust-region methods), combinatorial optimization, robust optimization

Based on: **[IPCO 19]**, **[Math. Prog. 20]**
Related: Yakubovich [1971], Yıldıran [2009], Ho-Nguyen and Kılınç-Karzan [2017]

# Example: QCQPs with symmetry

- Convex hull exactness in the case of "highly symmetric" QCQPs
- Suppose $A_{\text{obj}} = I_k \otimes \mathbb{A}_{\text{obj}}$, $A_i = I_k \otimes \mathbb{A}_i$ for all $i \in [m]$

$$I_k \otimes \mathbb{A} = \begin{pmatrix} \mathbb{A} & & & \\ & \mathbb{A} & & \\ & & \ddots & \\ & & & \mathbb{A} \end{pmatrix}$$

  and $k \geq m$
- **Applications**:
  - Robust least squares, sphere packing, QCQPs with spherical constraints, orthogonal Procrustes problem

Based on: **[Math. Prog. *under review*]**
Related: Beck [2007], Beck et al. [2012]

# Example: QCQPs with sign-definite linear terms

- Objective value exactness in the case of diagonal, sign-definite QCQPs
- Suppose $A_{\text{obj}}, \ldots, A_m$ diagonal and

$$\forall j \in [n], \left\{(b_{\text{obj}})_j, (b_1)_j, \ldots, (b_m)_j\right\} \text{ have the same sign}$$

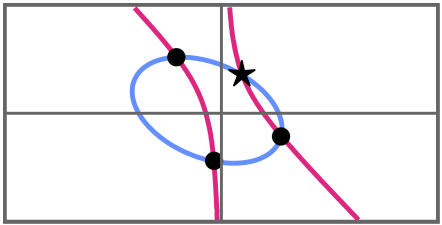
- Obj. val. exactness for

$$\min_{x \in \mathbb{R}^n} \left\{ x^\top A_{\text{obj}} x + 2 b_{\text{obj}}^\top x + c_{\text{obj}} : \|x\|_2 \leq 1, \|x\|_\infty \leq \alpha \right\}$$

Based on: **[Tut. Oper. Res. 21]**
Related: Burer and Ye [2019], Sojoudi and Lavaei [2014]

# Example: Random underconstrained quadratic systems

- Obj. val. exactness in the case of random underconstrained quadratic systems
- Solve $$\inf_{x \in \mathbb{R}^n} \left\{ \|x\|^2 : \; q_i(x) = 0, \; \forall i \in [m] \right\}$$



- Fix $m$, let $n \to \infty$, if data generated "as Gaussians", then objective value exactness w.p. $1 - o(1)$ (in fact, optimizer exactness)

Based on: **[Math. Prog. *under review*]**
Related: Burer and Ye [2019], Locatelli [2020]

# Summary of Part 1

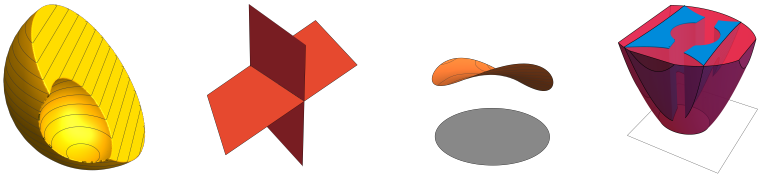
- Sufficient conditions for convex hull exactness
- **Necessary and sufficient** if $\Gamma$ is polyhedral (dual facially exposed)
- Sufficient conditions for objective value exactness
- Rank-one-generated (ROG) cones: QCQP-SDP analogue of integral polyhedra
- **Applications**:
  - Semi-random QCQPs, ratios of quadratic functions

| Exactness | ROG Cones |
| --- | --- |
| [IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*] | [Tut. Oper. Res. 21], [Math. Oper. Res. 21] |

## Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

**SDPs provide exact reformulations for broad classes of QCQPs!**



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# Revisiting the SDP relaxation

- Usual SDP relaxation in $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n \implies \Theta(n^2)$ variables
  - Interior point method $\implies \Theta\left(n^2 + m^2\right)$ storage
- Can we solve SDP more efficiently/using less storage if it is exact?
- Our view:

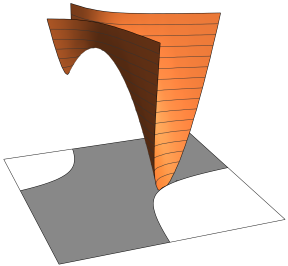
$$\mathrm{Opt}_{\mathsf{SDP}} \coloneqq \inf_{x \in \mathbb{R}^n} \left( \sup_{\gamma \in \Gamma} q(\gamma, x) \right)$$

  is a minimization problem in the original space $\implies n$ variables

# The generalized trust-region subproblem (GTRS)

- Special setting with single constraint ($\leq$ or $=$)

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \leq 0\}$$



- **TRS Applications**:
  - nonlinear programming (trust-region methods), combinatorial optimization, robust optimization

- **GTRS Applications**:
  - minimizing quartics of the form $q(x, p(x))$

$$\inf_{x \in \mathbb{R}^n, \alpha} \{q(x, \alpha) : \alpha = p(x)\}$$

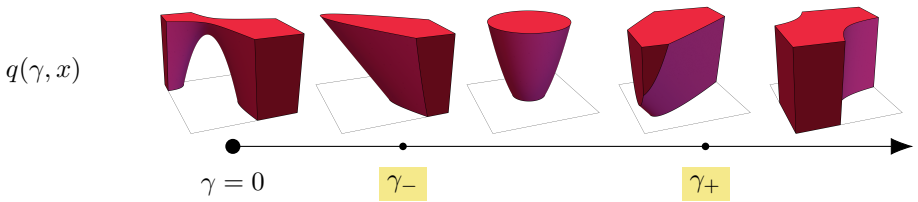
(source localization, constrained rank-one approximation), regression with adversarial data, iterative QCQP solvers

Based on: **[Math. Prog. 20]**, **[SIAM. J. Optim. *under review*]**, **[ICML *under review*]**

# Linear-time algorithm for the GTRS

- Convex hull exactness holds $\quad \operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}} = \left\{ (x,t) : \sup_{\gamma \in \Gamma} q(\gamma, x) \leq t \right\}$

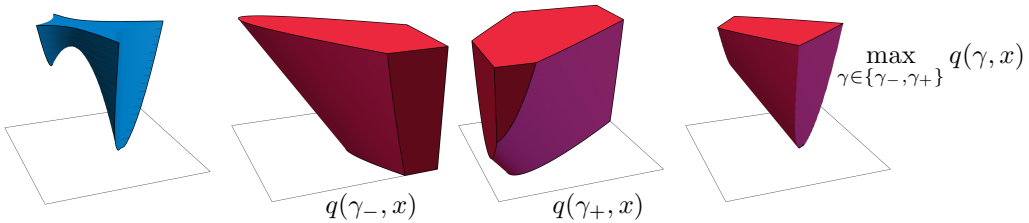
- Recall $\quad \Gamma = \{\gamma \in \mathbb{R}_+ : q(\gamma, x) \text{ is convex in } x\}$




Based on: **[Math. Prog. 20]**
Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

# Linear-time algorithm for the GTRS

- $\Gamma = [\gamma_-, \gamma_+] \implies \mathrm{Opt} = \mathrm{Opt}_{\mathrm{SDP}} = \inf_{x \in \mathbb{R}^n} \max_{\gamma \in \{\gamma_-, \gamma_+\}} q(\gamma, x)$




- **Algorithmic idea**
  - Compute $\gamma_-$ and $\gamma_+$ to some accuracy
  - Apply accelerated gradient descent

$$\implies \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \frac{1}{\sqrt{\epsilon}}$$

Based on: **[Math. Prog. 20]**
Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

# Is this running time optimal?

- GTRS:

$$\inf_{x \in \mathbb{R}^n} \left\{ q_{\text{obj}}(x) : q_1(x) \leq 0 \right\} \qquad \tilde{O}\left( \frac{N}{\sqrt{\epsilon}} \log\left( \frac{n}{p} \right) \log\left( \frac{1}{\epsilon} \right) \right)$$

- Minimum eigenvalue:

$$\inf_{x \in \mathbb{R}^n} \left\{ x^\mathsf{T} A x : \|x\|^2 = 1 \right\} \qquad O\left( \frac{N}{\sqrt{\epsilon}} \log\left( \frac{n}{p} \right) \right)$$

- Smooth convex quadratic:

$$\inf_{x \in \mathbb{R}^n} q_{\text{obj}}(x) \qquad O\left( \frac{N}{\sqrt{\epsilon}} \right)$$

- But, these are hardest problems within the GTRS!
- Can do better when regularity $\mu^* > 0$

Based on: **[Math. Prog. 20]**
Related: Kuczynski and Wozniakowski [1992], Nesterov [2018]

# Linear convergence for regular GTRS

- $\mu^* > 0$ holds for most GTRS
- Dual problem

$$\mathrm{Opt}_{\mathsf{SDP}} := \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x) = \sup_{\gamma \in \Gamma} \inf_{x \in \mathbb{R}^n} q(\gamma, x)$$

## Definition

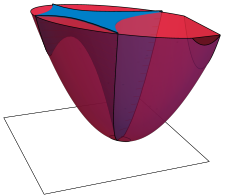
Let $\gamma^*$ be dual optimizer. Define $\mu^* := \lambda_{\min}(A_{\mathsf{obj}} + \gamma^* A_1)$. GTRS instance is regular if $\mu^* > 0$.

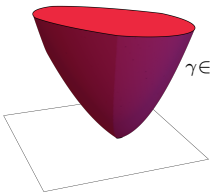
- Minimum eigenvalue problem is not regular
- Some instances of smooth quadratic minimization are not regular

Based on: **[SIAM J. Optim. *under review*]**
Related: Carmon and Duchi [2018]

# Linear convergence for regular GTRS

- Suppose $\mu^* > 0$





$$\max_{\gamma \in \{\tilde{\gamma}_-, \tilde{\gamma}_+\}} q(\gamma, x)$$

- Suppose $\gamma^* \in [\tilde{\gamma}_-, \tilde{\gamma}_+] \subseteq \Gamma \implies \mathrm{Opt} = \inf_{x \in \mathbb{R}^n} \max_{\gamma \in \{\tilde{\gamma}_-, \tilde{\gamma}_+\}} q(\gamma, x)$
- Suffices to estimate $\gamma^*$ roughly and can exploit strong convexity

$$\tilde{O}\left(\frac{N}{\sqrt{\mu^*}} \log\left(\frac{1}{\mu^*}\right) \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \log\left(\frac{1}{\epsilon}\right)$$

- Linear in $N$ and $\log(1/\epsilon)$

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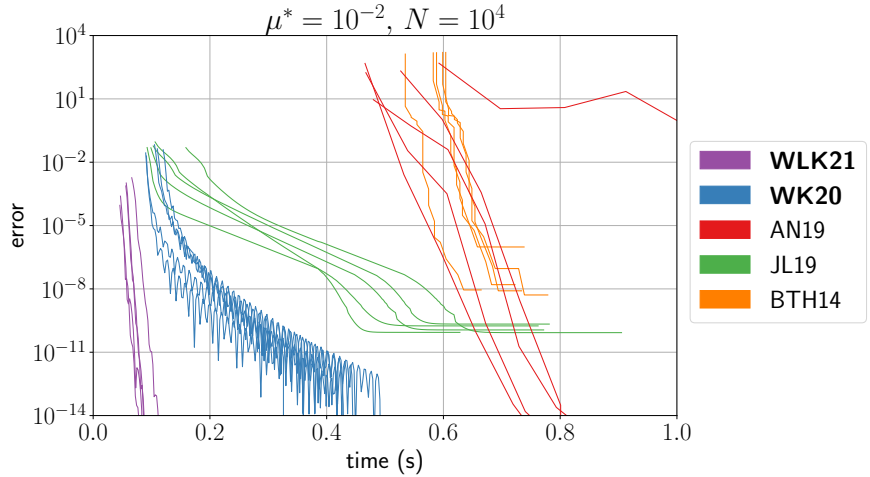
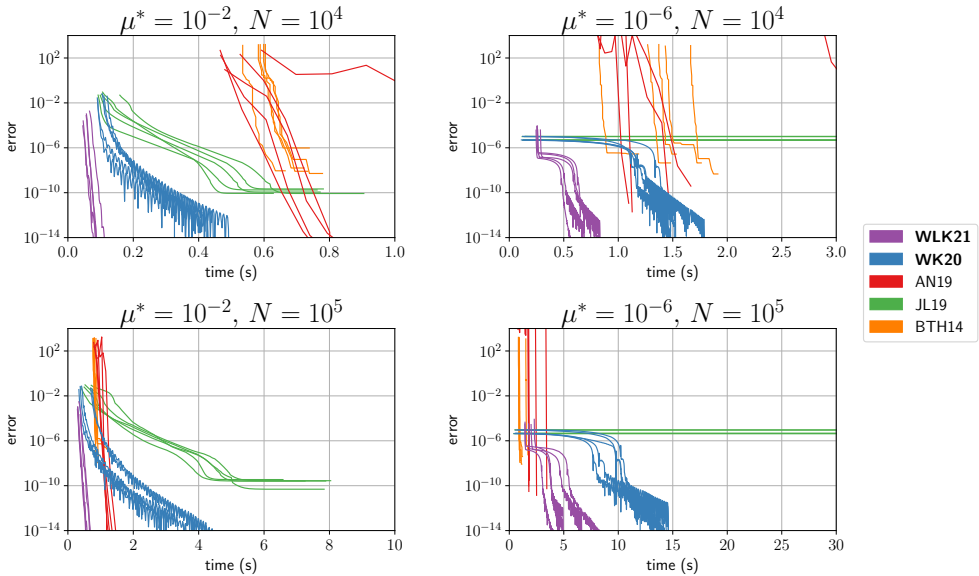
Based on: **[SIAM J. Optim. *under review*]**
Related: Carmon and Duchi [2018]

# Numerical experiments, n = 1,000

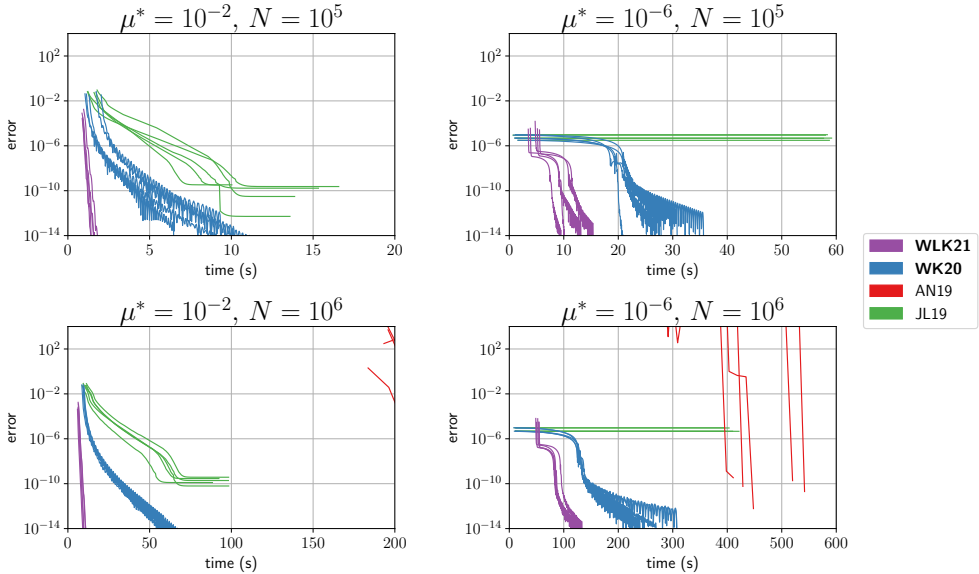



Based on: **[Math. Prog. 20]**, **[SIAM. J. Optim. *under review*]**
Related: Ben-Tal and den Hertog [2014], Jiang and Li [2019], Adachi and Nakatsukasa [2019]

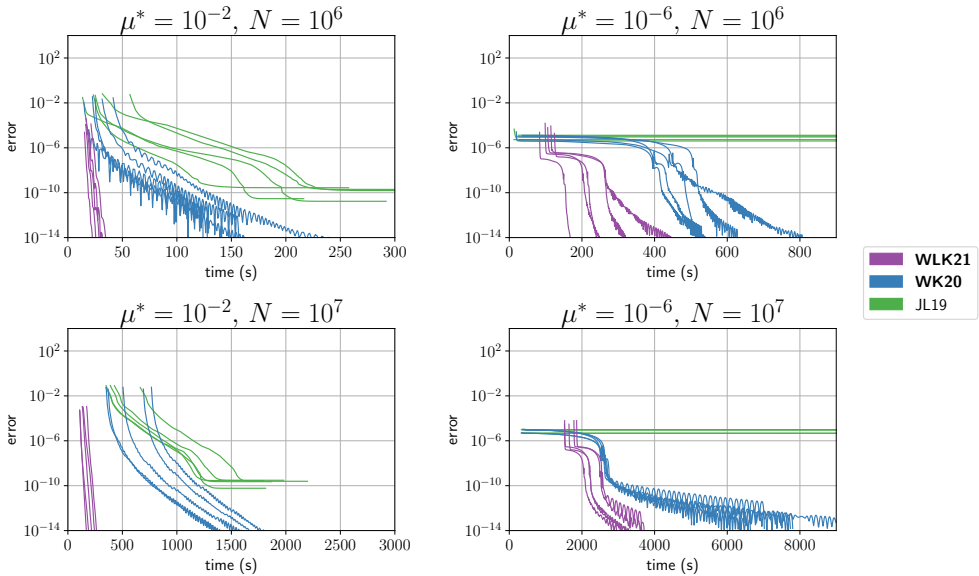
# Numerical experiments, n = 1,000



# Numerical experiments, n = 10,000



# Numerical experiments, n = 100,000



# Efficient algorithms for regular QCQPs

- Regularity can also be defined for general QCQPs! (low-rank SDPs)

## Definition

Let $\gamma^*$ be dual optimizer. Define $\mu^* := \lambda_{\min}\left(A_{\text{obj}} + \sum_{i=1}^m \gamma_i^* A_i\right)$. QCQP instance is regular if $\mu^* > 0$.

- $\mu^* > 0 \implies$ objective value exactness
- $\mu^* > 0$ in a number of statistical recovery problems: phase-retrieval, clustering
- Contributions
  - **Conceptual**: If $\mu^* > 0$, then can construct strongly convex reformulation in time independent of $\epsilon$
  - **Algorithmic**: Can solve strongly convex reformulation efficiently $\Longleftarrow$ grad-map
  - **Practical**: Preliminary implementation and numerical experiments

Based on: **[Ongoing]**

# Constructing a strongly convex reformulation of regular QCQP

- Consider a subgradient method for **dual problem**

$$\sup_{\gamma \in \Gamma} \inf_{x \in \mathbb{R}^n} q(\gamma, x)$$

- Iterates $\gamma^{(1)}, \gamma^{(2)}, \ldots$ and bounds $\left\|\gamma^{(i)} - \gamma^*\right\| \leq \rho^{(i)}$
- Will stop once $B(\gamma^{(t)}, \rho^{(t)}) \subseteq \operatorname{int}(\Gamma)$
- Then, $$\operatorname{Opt} = \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in B(\gamma^{(t)}, \rho^{(t)})} q(\gamma, x)$$
- Independent of $\epsilon$, can be done with $O(m+n)$ storage

Based on: **[ongoing]**
Related: Ding et al. [2021]

# Preliminary numerical experiments

- 10 synthetic instances: $n = 5000$, $m = 50$, density $= 0.001$, $\mu^* = 0.1$
- Primal-dual solver

| Section | ncalls | avg time | %tot | avg error |
|---|---|---|---|---|
| Total | 10 | 1740s | 100% | |
| dual_solve | 10 | 1691s | 97.2% | 1.15e-02 |
| eigsolve | 33.1k | 506ms | 96.0% | |
| primal_solve | 10 | 48.7s | 2.80% | 3.61e-12 |

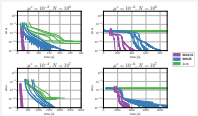
- Splitting Conic Solver (SCS): avg time 18150s

# Summary of Part 2

- Efficient algorithms for the GTRS
- Efficient algorithms for regular QCQPs (low-rank SDPs)
- Algorithms for diagonalizing QCQPs

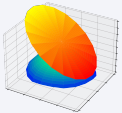
**GTRS**

[Math. Prog. 20],
[SIAM J. Optim. *under review*],
[ICML *under review*]



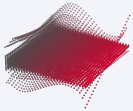
**Regular QCQPs**

[Ongoing]



**Diagonalizing QCQPs**

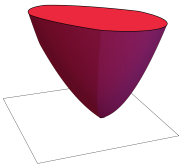
[Math. Prog. *under review*]

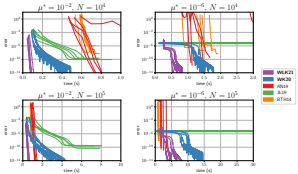


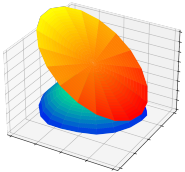
## Long-term research goal

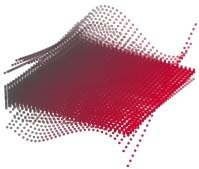
Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

**Some nonconvex problems can be solved efficiently via first-order methods!**







1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

4 Conclusion and future directions

# Future work

## Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Solving nonconvex problems accurately**
  - **Completed**: SDPs provide exact reformulations for broad classes of QCQPs!
  - **Future**:
    - Can we understand approximation quality systematically within general framework?
    - Can we understand exactness/approximation for more powerful convex relaxations?

# Future work

## Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Solving nonconvex problems efficiently**
  - **Completed**: Some nonconvex problems can be solved efficiently via first-order methods!
  - **Future**:
    - Exactness $\approx$ efficiency?
    - Can we develop efficient algorithms for semidefinite programs with low-rank solutions
    - Can we approximate "expensive" tools (e.g., SDPs) with cheap tools (e.g., linear programs, second-order cone programs)

# Summary of my research

## Exactness

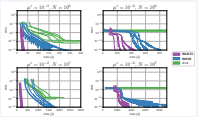
[IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*]

## ROG Cones

[Tut. Oper. Res. 21], [Math. Oper. Res. 21]

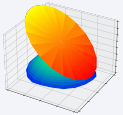
## GTRS

[Math. Prog. 20], [SIAM J. Optim. *under review*], [ICML *under review*]



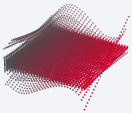
## Regular QCQPs

[Ongoing]



## Diagonalizing QCQPs

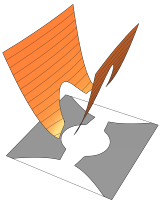
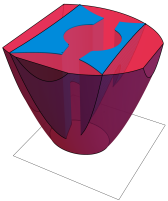
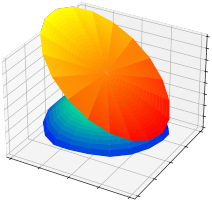
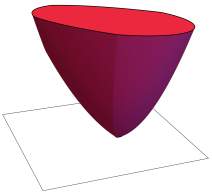
[Math. Prog. *under review*]



## Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

**Thank you! Questions?**

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# Rank-one-generated cones

## Definition

Cone $\mathcal{S} \subseteq \mathbb{S}^n_+$ is rank-one-generated (ROG) if $\mathcal{S} = \operatorname{conv}\left(\mathcal{S} \cap \left\{xx^\top\right\}\right)$.

Compare: $P \subseteq [0,1]^n$ is integral if $P = \operatorname{conv}(P \cap \{0,1\}^n)$

- Given QCQP, if constraints correspond to ROG cone, then objective value exactness and convex hull exactness regardless of objective function
- Suppose $\mathcal{S} = \left\{X \in \mathbb{S}^n_+ : \langle M, X\rangle \leq 0, \forall M \in \mathcal{M}\right\}$

## Goal

What properties of $\mathcal{M} = \{M_1, \ldots, M_k\}$ imply $\mathcal{S}$ is ROG?

Based on: **[Math. Oper. Res. 21]**, **[Tut. Oper. Res. 21]**

**Theorem (Sufficient conditions)**

$\mathcal{S}$ is ROG if

- for all $i \neq j$, there exists $(\alpha, \beta) \neq (0, 0)$ such that $\alpha M_i + \beta M_j \succeq 0$, or
- there exists $a \in \mathbb{R}^n$ such that $M_i = ab_i^\top + b_i a^\top$.

**Theorem (Characterization of ROG for $|\mathcal{M}| = 2$)**

Suppose $\mathcal{M} = \{M_1, M_2\}$. Then sufficient condition above is also necessary.

Based on: **[Math. Oper. Res. 21]**, **[Tut. Oper. Res. 21]**

# ROG and ratios of quadratic functions

- $$\inf_{z\in\mathbb{R}^{n+1}} \left\{ \frac{z^\top M_{\text{obj}} z}{z^\top B z} : \begin{array}{l} z^\top M_i z \le 0,\ \forall i \in [m] \\ z^\top B z > 0 \\ z_{n+1}^2 = 1 \end{array} \right\}$$
$$= \inf_{\tilde{z}\in\mathbb{R}^{n+1}} \left\{ \tilde{z}^\top M_{\text{obj}} \tilde{z} : \begin{array}{l} \tilde{z}^\top M_i \tilde{z} \le 0,\ \forall i \in [m] \\ \tilde{z}^\top B \tilde{z} = 1 \\ \tilde{z}_{n+1}^2 > 0 \end{array} \right\}$$
$$\ge \inf_{Z\in\mathbb{S}_+^{n+1}} \left\{ \langle M_{\text{obj}}, Z\rangle : \begin{array}{l} \langle M_i, Z\rangle \le 0,\ \forall i \in [m] \\ \langle B, Z\rangle = 1 \end{array} \right\}$$

- Equality holds if $\mathcal{S}(\{M_1, \ldots, M_m\})$ is ROG (+ minor assumptions)
- Example: Regularized total least squares

Based on: **[Math. Oper. Res. 21]**, **[Tut. Oper. Res. 21]**

# Stackelberg prediction games with least squares losses

- True data $(x_i, \alpha_i, \beta_i)_{i=1}^m$
- Leader (learner) chooses $w \in \mathbb{R}^n$
- Follower (data provider) modifies $x_i \to \tilde{x}_i$ so that $\langle w, \tilde{x}_i \rangle \approx \beta_i$
- Leader has loss $(\langle w, \tilde{x}_i \rangle - \alpha_i)^2$
- $$\min_{w \in \mathbb{R}^n} \left\{ \sum_{i=1}^m (\langle w, \tilde{x}_i \rangle - \alpha_i)^2 : \tilde{x}_i \in \arg\min_{x \in \mathbb{R}^n} \gamma \|x - x_i\|^2 + (\langle w, x \rangle - \beta_i)^2 \right\}$$

Based on: **[ICML *under review*]**

# Stackelberg prediction games with least squares losses

- $$\begin{aligned}
&\min_{w\in\mathbb{R}^n}\left\{\sum_{i=1}^m \left(\langle w,\tilde{x}_i\rangle-\alpha_i\right)^2 :\ \tilde{x}_i\in\operatorname*{arg\,min}_{x\in\mathbb{R}^n}\gamma\left\|x-x_i\right\|^2+\left(\langle w,x\rangle-\beta_i\right)^2\right\}\\
&\quad=\min_{w\in\mathbb{R}^n}\left\{\left\|\tilde{X}^\top w-\alpha\right\|^2 :\ \tilde{X}=(\gamma I+ww^\top)^{-1}(\gamma X+w\beta^\top)\right\}\\
&\quad=\min_{w\in\mathbb{R}^n}\left\|\frac{\|w\|^2\beta+\gamma Xw}{\|w\|^2+\gamma}-\alpha\right\|^2\\
&\quad=\min_{w\in\mathbb{R}^n,t\in\mathbb{R}}\left\{\left\|\frac{t\beta+\gamma Xw}{t+\gamma}-\alpha\right\|^2 :\ t=\|w\|^2\right\}\\
&\quad=\min_{\tilde{w}\in\mathbb{R}^n,\tilde{t}\in\mathbb{R}}\left\{\left\|\frac{\tilde{t}}{2}\beta+\frac{\sqrt{\gamma}}{2}X\tilde{w}-\left(\alpha-\frac{\beta}{2}\right)\right\|^2 :\ \|\tilde{w}\|^2+\tilde{t}^2=1\right\}
\end{aligned}$$

Based on: **[ICML *under review*]**

# Diagonalizing QCQPs

## Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is simultaneously diagonalizable via congruence (SDC) if there exists invertible $P \in \mathbb{R}^{n \times n}$ such that $P^\top A_i P$ is diagonal $\forall i$.

- Nice property because: SDP relaxation of diagonal QCQP is SOCP (faster), $\Gamma$ is polyhedral (better understanding of exactness)

## Goal

Most sets of matrices are not SDC, can we find other computationally variants of SDC and understand such properties?

Based on: **[Math. Prog. *under review*]**

# Diagonalizing QCQPs: variants of SDC

## Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is almost SDC (ASDC) if for all $\epsilon > 0$, there exists $\|A_i' - A_i\| \leq \epsilon$ such that $\{A_i'\}$ is SDC.

- "Limit of SDC sets"

## Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is $d$-restricted SDC ($d$-RSDC) if there exists $A_i' = \left(\begin{smallmatrix} A_i & * \\ * & * \end{smallmatrix}\right) \in \mathbb{S}^{n+d}$ such that $\{A_i'\}$ is SDC.

- "Restriction of SDC sets"

Based on: **[Math. Prog. *under review*]**

# Diagonalizing QCQPs: ASDC

**Theorem**

Let $\{A, B\} \subseteq \mathbb{S}^n$ and suppose $A$ invertible. Then $\{A, B\}$ is ASDC if and only if $A^{-1}B$ has real spectrum. (+ construction)

**Theorem**

Let $\{A, B\} \subseteq \mathbb{S}^n$. If $\{A, B\}$ is singular, then it is ASDC. (+ construction)

**Theorem**

Let $\{A, B, C\} \subseteq \mathbb{S}^n$ and suppose $A$ invertible. Then $\{A, B, C\}$ is ASDC if and only if $\left\{A^{-1}B, A^{-1}C\right\}$ commute and have real spectrum. (+ construction)

Based on: **[Math. Prog. *under review*]**

# Diagonalizing QCQPs: $d$-RSDC

**Theorem**

Let $\{A, B\} \subseteq \mathbb{S}^n$. If $A$ is invertible and $A^{-1}B$ has simple eigenvalues, then $\{A, B\}$ is $1$-RSDC. (+ construction)

- Condition holds generically

Based on: **[Math. Prog. *under review*]**

# “Fuzzy” spectral partitioning

- Connected graph $G = (V, E)$
- Vertex masses $\mu : V \to \mathbb{R}_{++}$ and edge weights $\kappa : E \to \mathbb{R}_{++}$
- Laplacian $L = D - A$ w.r.t. $\kappa$

**Theorem (Cheeger’s inequality)**

If $\mu_v = d_v$, then

$$\frac{\Phi^2}{2} \leq \lambda_2(L, M) \leq 2\Phi$$

- $\lambda_2(L, M)$ is first nontrivial generalized eigenvalue
- $\Phi$ is sparsest cut

Based on: **[APPROX 19]**

# “Fuzzy” spectral partitioning

- We define “Fuzzy cuts”

## Definition

$$\Psi \approx \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\min\left(\mu(A), \mu(B)\right)},\ A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$

- $\Phi$ must partition, $\Psi$ may leave out. $\Psi = \Phi$ if $A, B$ is a partition.

## Theorem

$$\frac{\Psi}{4} \leq \lambda_2(L, M) \leq \Psi$$

Based on: **[APPROX 19]**

# Stable Euclidean $k$-means

- $k$-means clustering: $\{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$
- Suppose there exist true clustering that is unique optimum even if for all $i$, $x_i \mapsto x_i' \in B(x_i, \epsilon)$

## Theorem

**Two clusters.** There exists $c \geq 1$ such that for any fixed $\epsilon > 0$, we can recover true clustering in time $d \cdot n^{O(\epsilon^{-c})}$.

Additional results for $\geq 3$ clusters given an additional "separation" assumption

Based on: **[NeurIPS 17]**