Sufficient Conditions for Exact SDP Reformulations of QCQPs

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Quadratically Constrained Quadratic Programs (QCQP)

- $q_0, q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ be (possibly nonconvex!) quadratic functions

$$q_i(x) = x^\top A_i x + 2b_i^\top x + c_i$$

- Want to find

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \left\{ q_0(x) \;\middle|\; \begin{array}{l} q_1(x) \le 0 \\ \vdots \\ q_m(x) \le 0 \end{array} \right\}$$

The QCQP Epigraph

$$\mathrm{Opt} = \inf_{x\in\mathbb{R}^n} \{q_0(x) \,|\, q_i(x) \le 0,\ \forall i \in [m]\}$$
$$= \inf_{x,t} \left\{ t \;\middle|\; \begin{array}{l} q_0(x) \le t \\ q_i(x) \le 0,\ \forall i \in [m] \end{array} \right\} =: \inf_{x,t} \left\{ t \;\middle|\; (x,t) \in \mathcal{E} \right\}$$
$$= \inf_{x,t} \{t \,|\, (x,t) \in \mathrm{conv}(\mathcal{E})\}$$

The standard SDP relaxation of QCQP

Standard (Shor) SDP relaxation

$$\mathrm{Opt} = \inf_{x,X} \left\{ \langle A_0, X\rangle + 2b_0^\top x + c_0 \;\middle|\; \begin{array}{l} \langle A_1, X\rangle + 2b_1^\top x + c_1 \le 0 \\ \vdots \\ \langle A_m, X\rangle + 2b_m^\top x + c_1 \le 0 \\ X = xx^\top \end{array} \right\}$$

$$\ge \inf_{x,X} \left\{ \langle A_0, X\rangle + 2b_0^\top x + c_0 \;\middle|\; \begin{array}{l} \langle A_1, X\rangle + 2b_1^\top x + c_1 \le 0 \\ \vdots \\ \langle A_m, X\rangle + 2b_m^\top x + c_1 \le 0 \\ X \succeq xx^\top \end{array} \right\}$$

$$=: \mathrm{Opt}_{\mathsf{SDP}}$$

SDP epigraph

$$\text{Opt}_{\text{SDP}} = \inf_{x,X} \left\{ \langle A_0, X\rangle + 2b_0^\top x + c_0 \;\middle|\; \begin{array}{l} \langle A_1, X\rangle + 2b_1^\top x + c_1 \le 0 \\ \vdots \\ \langle A_m, X\rangle + 2b_m^\top x + c_1 \le 0 \\ X \succeq xx^\top \end{array} \right\}$$

$$= \inf_{x,X,t} \left\{ t \;\middle|\; \begin{array}{l} \langle A_0, X\rangle + 2b_0^\top x + c_0 \le t \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0,\ \forall i \in [m] \\ X \succeq xx^\top \end{array} \right\}$$

- Let $\mathcal{E}_{\text{SDP}}$ be the projection of the epigraph onto (x, t) variables

The story so far

- QCQP is NP hard, but we can solve SDP relaxation instead
- What are sufficient conditions for
  - SDP tightness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$?
  - Convex hull result: $\mathrm{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}}$?

Related work

- SDP reformulations for QCQPs with a single constraint
  [Fradkov and Yakubovich, 1979], [W, Kılınç-Karzan, 2019]
- SDP reformulations for variants of TRS
  [Sturm and Zhang, 2003], [Burer, 2015], [Yang, Anstreicher, and Burer, 2018]
- SDP reformulations for Diagonal QCQPs
  [Burer and Ye, 2018]
- Quadratic Matrix Programming
  [Beck, 2007], [Beck, Drori, and Teboulle, 2012]

Outline

- Analyze a parameter k which captures “amount of symmetry” in a given QCQP
- Under additional “polyhedrality” assumption, $k \geq \operatorname{aff}\dim(\{b_i\}) + 1 \implies$ SDP tightness and convex hull result

Quadratic eigenvalue multiplicity

Definition

Let $1 \le k \le n$ be the largest integer such that for each $i = 0, \ldots, m$, the matrix $A_i \in \mathbb{S}^n$ has the following block form

$$A_i = \mathcal{A}_i \otimes I_k = \begin{pmatrix} \mathcal{A}_i & & & \\ & \mathcal{A}_i & & \\ & & \ddots & \\ & & & \mathcal{A}_i \end{pmatrix}$$

where $\mathcal{A}_i \in \mathbb{S}^{n/k}$

Quadratic eigenvalue multiplicity

- $A_i = \mathcal{A}_i \otimes I_k$
- Suppose $n = 4$

$x_1^2 + x_2^2 + x_3^2 + x_4^2$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$k = 4$
$(x_1 - x_2)^2 + (x_3 - x_4)^2$	$\begin{pmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & -1 & 1 \end{pmatrix}$	$k = 2$

Our first result

Theorem

Suppose primal feasibility and dual strict feasibility. If Γ is polyhedral and

$$k \geq \operatorname{aff}\dim(\{b(\gamma) \mid \gamma \in \mathcal{F}\}) + 1$$

for every semidefinite face $\mathcal{F}$ of Γ of affine dimension at most $m-1$, then

$$\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\text{SDP}} \quad \text{and} \quad \operatorname{Opt} = \operatorname{Opt}_{\text{SDP}}.$$

$$\Gamma := \left\{ \gamma \in \mathbb{R}^m \;\middle|\; \gamma \geq 0,\ A_0 + \sum_{i=1}^{m} \gamma_i A_i \succeq 0 \right\}$$

Our first result

Corollary

Suppose primal feasibility and dual strict feasibility. If $\{A_i\}_{i=0}^m$ are diagonal and

$$k \geq \min\left(m\ ,\ 1 + |\{b_i \neq 0\}_{i=1}^m|\right),$$

then

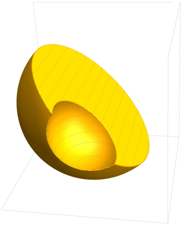
$$\mathrm{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}} \quad \text{and} \quad \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}\,.$$

- When $m = 1$
- When $b_1 = b_2 = \cdots = b_m = 0$, the condition is trivially satisfied
- $A_i = \alpha_i I_n$ for all $i = 0, \ldots, m$ and $n \geq m$

Example: Swiss cheese

- $A_i = \alpha_i I_n$ for all $i = 0, \ldots, m$ and $n \geq m$
- Minimizing distance to a piece of Swiss cheese

$$\inf_{x \in \mathbb{R}^n} \left\{ \|x\|^2 \;\middle|\; \begin{array}{l} \text{inside ball constraints} \\ \text{outside ball constraints} \\ \text{linear constraints} \end{array} \right\}$$



- If nonempty and $n \geq m$, then the standard SDP relaxation is tight for this QCQP

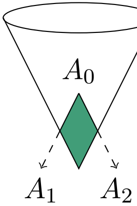
Aggregation

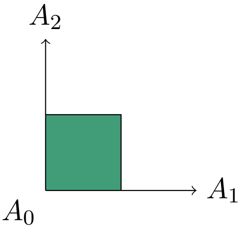
- For $\gamma \in \mathbb{R}^m$ such that $\gamma \geq 0$, define

$$q(\gamma, x) := q_0(x) + \sum_{i=1}^{m} \gamma_i q_i(x) \qquad A(\gamma) := A_0 + \sum_{i=1}^{m} \gamma_i A_i$$

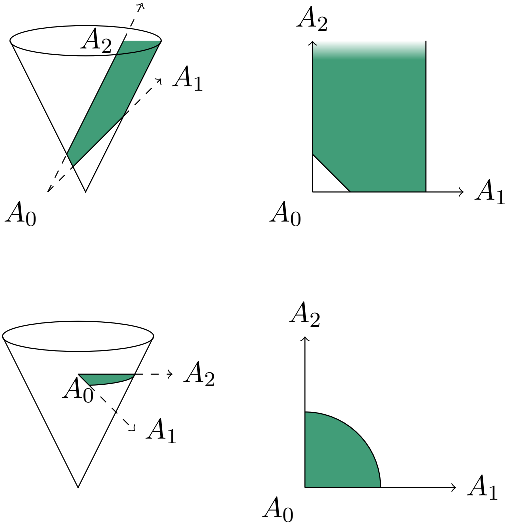
- Define

$$\Gamma := \{\gamma \in \mathbb{R}^m : \gamma \geq 0, \, A(\gamma) \succeq 0\}$$





What does Γ look like?



Rewriting the SDP in terms of Γ

- Γ plays a crucial role in the analysis!

Theorem

Suppose primal feasibility and dual strict feasibility, then

$$\mathrm{Opt}_{\mathsf{SDP}} = \min_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q(\gamma, x)$$

$$\mathcal{E}_{\mathsf{SDP}} = \left\{ (x,t) : \sup_{\gamma\in\Gamma} q(\gamma, x) \leq t \right\}$$

- If Γ is polyhedral, then $\mathcal{E}_{\mathsf{SDP}}$ is defined by finitely many convex quadratics $\implies$ SOC-representable

Conclusion

- Analyzed
  - "amount of symmetry" k
  - the geometry of a dual object Γ
- Assuming primal feasibility and dual strict feasibility

	Polyhedral Γ	General Γ
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) \mid \gamma \in \mathcal{F}\}) + 1$	$m + 1$
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) \mid \gamma \in \mathcal{F}\}) + 1$	$m + 2$

- Results extend to equalities!
- A general framework for proving sufficient conditions
- Future directions
  - Are the assumptions on k sharp?
  - Can techniques say anything about when SDP is approximately tight?

Thank you. Questions?

Slides and preprint available (hopefully) soon

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