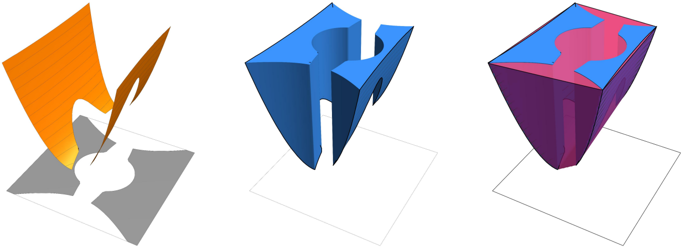
On Convex Hulls of Epigraphs of QCQPs

Alex L. Wang Fatma Kılınç-Karzan

Carnegie Mellon University

IPCO 2020



Supported in part by NSF grant CMMI 1454548 and ONR grant N00014-19-1-2321.

Quadratically Constrained Quadratic Programs (QCQP)

- $q_0, q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ (possibly nonconvex!) quadratic functions

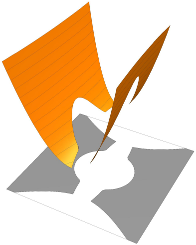
$$q_i(x) = x^\top A_i x + 2b_i^\top x + c_i$$

Quadratically Constrained Quadratic Programs (QCQP)

- $q_0, q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ (possibly nonconvex!) quadratic functions
$$q_i(x) = x^\top A_i x + 2b_i^\top x + c_i$$
- Want to find
$$\boxed{\text{Opt}} := \inf_{x \in \mathbb{R}^n} \left\{ q_0(x) : \begin{array}{c} q_1(x) \le 0 \\ \vdots \\ q_m(x) \le 0 \end{array} \right\}$$

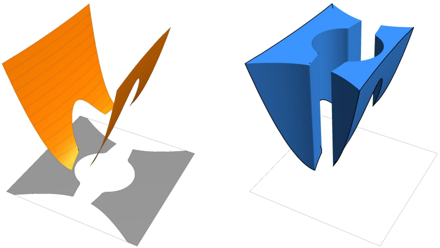
The QCQP Epigraph

$$\text{Opt} = \inf_{x \in \mathbb{R}^n} \{q_0(x) : q_i(x) \leq 0, \forall i \in [m]\}$$



The QCQP Epigraph

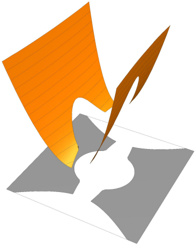
$$\begin{aligned}\mathrm{Opt} &= \inf_{x\in\mathbb{R}^n} \{q_0(x) :\ q_i(x) \le 0,\ \forall i \in [m]\} \\ &= \inf_{x,t} \left\{ t :\ \begin{array}{l} q_0(x) \le t \\ q_i(x) \le 0,\ \forall i \in [m] \end{array} \right\}\end{aligned}$$

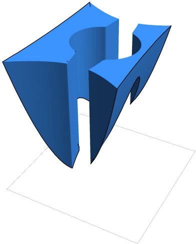


The QCQP Epigraph

$$\text{Opt} = \inf_{x\in\mathbb{R}^n} \{q_0(x) : \; q_i(x) \le 0, \, \forall i \in [m]\}$$

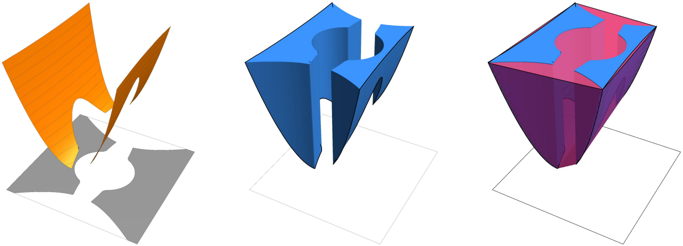
$$= \inf_{x,t} \left\{ t : \; \begin{array}{l} q_0(x) \le t \\ q_i(x) \le 0, \, \forall i \in [m] \end{array} \right\} =: \inf_{x,t} \left\{ t : \; (x,t) \in \mathcal{E} \right\}$$





The QCQP Epigraph

$$\begin{aligned}
\mathrm{Opt} &= \inf_{x\in\mathbb{R}^n} \{q_0(x) :\ q_i(x) \le 0,\ \forall i \in [m]\} \\
&= \inf_{x,t} \left\{ t :\ \begin{array}{l} q_0(x) \le t \\ q_i(x) \le 0,\ \forall i \in [m] \end{array} \right\} =: \inf_{x,t} \left\{ t :\ (x,t) \in \mathcal{E} \right\} \\
&= \inf_{x,t} \{ t :\ (x,t) \in \mathrm{conv}(\mathcal{E}) \}
\end{aligned}$$



The standard SDP relaxation of QCQP

Standard (Shor) SDP relaxation

$$\mathrm{Opt} = \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \leq 0, \ \forall i \\ Y = xx^\top \end{array} \right\}$$

The standard SDP relaxation of QCQP

Standard (Shor) SDP relaxation

$$\begin{aligned}\mathrm{Opt} &= \inf_{x,Y}\left\{\langle A_0, Y\rangle + 2b_0^\top x + c_0 : \begin{array}{l}\langle A_i, Y\rangle + 2b_i^\top x + c_i \le 0,\ \forall i\\ Y = xx^\top\end{array}\right\}\\ &\ge \inf_{x,Y}\left\{\langle A_0, Y\rangle + 2b_0^\top x + c_0 : \begin{array}{l}\langle A_i, Y\rangle + 2b_i^\top x + c_i \le 0,\ \forall i\\ Y \succeq xx^\top\end{array}\right\}\end{aligned}$$

The standard SDP relaxation of QCQP

Standard (Shor) SDP relaxation

$$
\begin{aligned}
\mathrm{Opt} &= \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \le 0,\ \forall i \\ Y = xx^\top \end{array} \right\} \\
&\ge \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \le 0,\ \forall i \\ Y \succeq xx^\top \end{array} \right\} \\
&=: \mathrm{Opt}_{\mathsf{SDP}}
\end{aligned}
$$

SDP epigraph

$$\mathrm{Opt}_{\mathsf{SDP}} = \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \leq 0,\ \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

SDP epigraph

$$\mathrm{Opt}_{\mathsf{SDP}} = \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \leq 0, \, \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

$$= \inf_{x,Y,t} \left\{ t : \begin{array}{l} \langle A_0, Y \rangle + 2b_0^\top x + c_0 \leq t \\ \langle A_i, Y \rangle + 2b_i^\top x + c_i \leq 0, \, \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

SDP epigraph

$$\mathrm{Opt}_{\mathsf{SDP}} = \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \le 0,\ \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

$$= \inf_{x,Y,t} \left\{ t : \begin{array}{l} \langle A_0, Y \rangle + 2b_0^\top x + c_0 \le t \\ \langle A_i, Y \rangle + 2b_i^\top x + c_i \le 0,\ \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

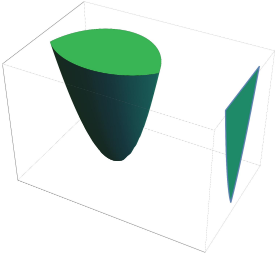
- Let $\mathcal{E}_{\mathsf{SDP}}$ be projection of SDP epigraph onto (x, t) variables

SDP epigraph

$$\mathrm{Opt}_{\mathsf{SDP}} = \inf_{x,Y} \left\{ \langle A_0, Y \rangle + 2b_0^\top x + c_0 : \begin{array}{l} \langle A_i, Y \rangle + 2b_i^\top x + c_i \le 0, \ \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

$$= \inf_{x,Y,t} \left\{ t : \begin{array}{l} \langle A_0, Y \rangle + 2b_0^\top x + c_0 \le t \\ \langle A_i, Y \rangle + 2b_i^\top x + c_i \le 0, \ \forall i \\ Y \succeq xx^\top \end{array} \right\}$$

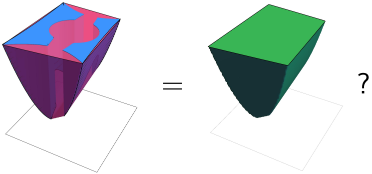
- Let $\mathcal{E}_{\mathsf{SDP}}$ be projection of SDP epigraph onto (x, t) variables

(x, Y, t)  (x, t)

The motivating question

What are sufficient conditions for

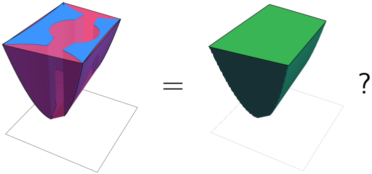
- Convex hull result: $\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}}$?



The motivating question

What are sufficient conditions for

- Convex hull result: $\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}}$?



- SDP tightness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$?

Related work

- Convex hull results

Related work

- Convex hull results
  - Single nonconvex quadratic $\cap$ additional constraints

    [Fradkov and Yakubovich 1979], [Yıldıran 2009], [Burer 2015], [Burer and Kılınç-Karzan 2017], [Santana and Dey 2018], [Yang, Anstreicher, and Burer 2018], [**W** and Kılınç-Karzan 2019] ...

Related work

- Convex hull results
  - Single nonconvex quadratic $\cap$ additional constraints

    [Fradkov and Yakubovich 1979], [Yıldıran 2009], [Burer 2015], [Burer and Kılınç-Karzan 2017], [Santana and Dey 2018], [Yang, Anstreicher, and Burer 2018], [**W** and Kılınç-Karzan 2019] . . .
  - Mixed binary sets

    Extensive literature!

Related work

- Convex hull results
  - Single nonconvex quadratic ∩ additional constraints

    [Fradkov and Yakubovich 1979], [Yıldıran 2009], [Burer 2015], [Burer and Kılınç-Karzan 2017], [Santana and Dey 2018], [Yang, Anstreicher, and Burer 2018], [**W** and Kılınç-Karzan 2019] ...
  - Mixed binary sets

    Extensive literature!
- SDP tightness

Related work

- Convex hull results
  - Single nonconvex quadratic ∩ additional constraints

    [Fradkov and Yakubovich 1979], [Yıldıran 2009], [Burer 2015], [Burer and Kılınç-Karzan 2017], [Santana and Dey 2018], [Yang, Anstreicher, and Burer 2018], [**W** and Kılınç-Karzan 2019] ...
  - Mixed binary sets

    Extensive literature!
- SDP tightness
  - Diagonal QCQPs

    [Ben-Tal and den Hertog 2014], [Locatelli 2016], [Burer and Ye 2019], ...

Related work

- Convex hull results
  - Single nonconvex quadratic $\cap$ additional constraints
    [Fradkov and Yakubovich 1979], [Yıldıran 2009], [Burer 2015], [Burer and Kılınç-Karzan 2017], [Santana and Dey 2018], [Yang, Anstreicher, and Burer 2018], [**W** and Kılınç-Karzan 2019] ...
  - Mixed binary sets
    Extensive literature!
- SDP tightness
  - Diagonal QCQPs
    [Ben-Tal and den Hertog 2014], [Locatelli 2016], [Burer and Ye 2019], ...
  - Quadratic matrix programming
    [Beck 2007], [Beck, Drori, and Teboulle 2012], ...

Outline

- SDP relaxation in the x-space and (dual) convex Lagrange multipliers

Outline

- SDP relaxation in the x-space and (dual) convex Lagrange multipliers
- A notion of symmetry

Outline

- SDP relaxation in the x-space and (dual) convex Lagrange multipliers
- A notion of symmetry
- Informally:
  If "the geometry of some dual object is nice" and "amount of symmetry is large", then convex hull result and SDP tightness.

Aggregation

- For $\gamma \in \mathbb{R}^m_+$, define

$$q_\gamma(x) \coloneqq q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \dots$$

Aggregation

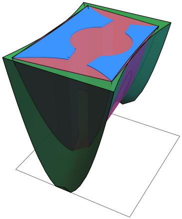
- For $\gamma \in \mathbb{R}^m_+$, define

$$q_\gamma(x) := q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \dots$$

- For all $\gamma \in \mathbb{R}^m_+$, $\quad (x,t) \in \mathcal{E} \implies q_\gamma(x) \le t$

Aggregation

- For $\gamma \in \mathbb{R}^m_+$, define
$$q_\gamma(x) := q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \dots$$
- For all $\gamma \in \mathbb{R}^m_+$, $\quad (x,t) \in \mathcal{E} \implies q_\gamma(x) \le t$

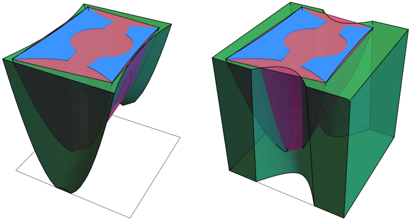


Aggregation

- For $\gamma \in \mathbb{R}^m_+$, define

$$q_\gamma(x) := q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \ldots$$

- For all $\gamma \in \mathbb{R}^m_+$, $\quad (x,t) \in \mathcal{E} \implies q_\gamma(x) \le t$



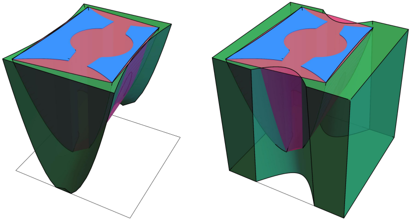
Aggregation

- For $\gamma \in \mathbb{R}^m_+$, define

$$q_\gamma(x) \coloneqq q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \dots$$

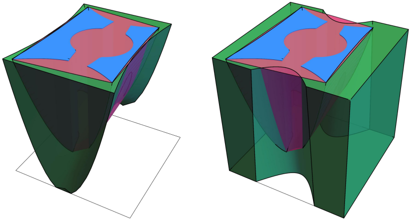
- For all $\gamma \in \mathbb{R}^m_+$, $\quad (x,t) \in \mathcal{E} \implies q_\gamma(x) \le t$
- Define

$$\Gamma \coloneqq \left\{ \gamma \in \mathbb{R}^m_+ : A_0 + \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}$$



Aggregation

- For $\gamma \in \mathbb{R}^m_+$, define
$$q_\gamma(x) := q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \dots$$
- For all $\gamma \in \mathbb{R}^m_+$, $\quad (x,t) \in \mathcal{E} \implies q_\gamma(x) \le t$
- Define
$$\Gamma := \left\{ \gamma \in \mathbb{R}^m_+ : A_0 + \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}$$
- For all $\gamma \in \Gamma$, $\quad (x,t) \in \operatorname{conv}(\mathcal{E}) \implies q_\gamma(x) \le t$



Aggregation

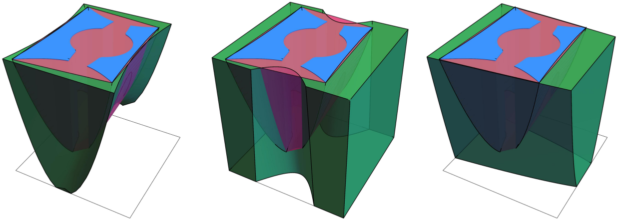
- For $\gamma \in \mathbb{R}^m_+$, define

$$q_\gamma(x) := q_0(x) + \sum_{i=1}^m \gamma_i q_i(x) = x^\top \left(A_0 + \sum_{i=1}^m \gamma_i A_i \right) x + \dots$$

- For all $\gamma \in \mathbb{R}^m_+$, $\quad (x,t) \in \mathcal{E} \implies q_\gamma(x) \le t$
- Define

$$\Gamma := \left\{ \gamma \in \mathbb{R}^m_+ : A_0 + \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}$$

- For all $\gamma \in \Gamma$, $\quad (x,t) \in \operatorname{conv}(\mathcal{E}) \implies q_\gamma(x) \le t$



Rewriting the SDP in terms of Γ

Theorem

Suppose primal feasibility and dual strict feasibility, then

$$\mathrm{Opt}_{\mathsf{SDP}} = \min_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q_\gamma(x)$$

$$\mathcal{E}_{\mathsf{SDP}} = \left\{ (x,t) : \sup_{\gamma\in\Gamma} q_\gamma(x) \le t \right\}$$

Rewriting the SDP in terms of Γ

Theorem

Suppose primal feasibility and dual strict feasibility, then

$$\mathrm{Opt}_{\mathsf{SDP}} = \min_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q_\gamma(x)$$

$$\mathcal{E}_{\mathsf{SDP}} = \left\{(x,t) : \sup_{\gamma\in\Gamma} q_\gamma(x) \le t\right\}$$

- When Γ is polyhedral, $\mathcal{E}_{\mathsf{SDP}}$ is defined by finitely many convex quadratics

What does Γ look like?

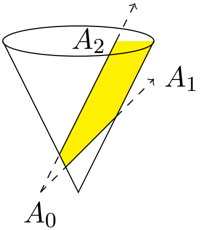
- Recall

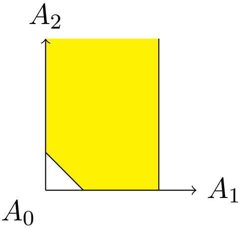
$$\Gamma = \left\{ \gamma \in \mathbb{R}^m_+ : \; A_0 + \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}$$

What does Γ look like?

- Recall

$$\Gamma = \left\{ \gamma \in \mathbb{R}^m_+ : \ A_0 + \sum_{i=1}^{m} \gamma_i A_i \succeq 0 \right\}$$

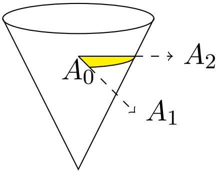


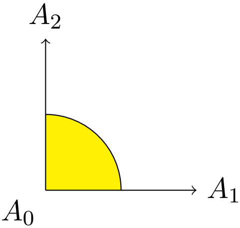


What does Γ look like?

- Recall

$$\Gamma = \left\{ \gamma \in \mathbb{R}^m_+ : \; A_0 + \sum_{i=1}^m \gamma_i A_i \succeq 0 \right\}$$





Quadratic eigenvalue multiplicity

Definition

Let $1 \leq k \leq n$ be the largest integer such that for each $i = 0, \ldots, m$, the matrix $A_i \in \mathbb{S}^n$ has the following block form

$$A_i = \hat{A}_i \otimes I_k = \begin{pmatrix} \hat{A}_i & & & \\ & \hat{A}_i & & \\ & & \ddots & \\ & & & \hat{A}_i \end{pmatrix}$$

where $\hat{A}_i \in \mathbb{S}^{n/k}$

Quadratic eigenvalue multiplicity

- $A_i = \hat{A}_i \otimes I_k$

Quadratic eigenvalue multiplicity

- $A_i = \hat{A}_i \otimes I_k$
- Suppose $n = 4$, i.e. $x \in \mathbb{R}^4$

Quadratic eigenvalue multiplicity

- $A_i = \hat{A}_i \otimes I_k$
- Suppose $n = 4$, i.e. $x \in \mathbb{R}^4$

$x_1^2 + x_2^2 + x_3^2 + x_4^2$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$k = 4$

Quadratic eigenvalue multiplicity

- $A_i = \hat{A}_i \otimes I_k$
- Suppose $n = 4$, i.e. $x \in \mathbb{R}^4$

$x_1^2 + x_2^2 + x_3^2 + x_4^2$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$k = 4$
$(x_1 - x_2)^2 + (x_3 - x_4)^2$	$\begin{pmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & -1 & 1 \end{pmatrix}$	$k = 2$

Our result

Corollary

Suppose primal feasibility and dual strict feasibility. If Γ is polyhedral and

$$k \geq \min\left(m, \left|\{b_i \neq 0\}_{i=1}^m\right| + 1\right),$$

then

$$\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}} \quad \text{and} \quad \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}.$$

Our result

Corollary

Suppose primal feasibility and dual strict feasibility. If Γ is polyhedral and

$$k \geq \min\left(m, \left|\{b_i \neq 0\}_{i=1}^{m}\right| + 1\right),$$

then

$$\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}} \quad \text{and} \quad \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}.$$

- $m = 1$

Our result

Corollary

Suppose primal feasibility and dual strict feasibility. If Γ is polyhedral and

$$k \geq \min\left(m, \ |\{b_i \neq 0\}_{i=1}^{m}| + 1\right),$$

then

$$\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}} \quad \text{and} \quad \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}\,.$$

- $m = 1$
- A_is diagonal and $b_1 = b_2 = \cdots = b_m = 0$

Our result

Corollary

Suppose primal feasibility and dual strict feasibility. If Γ is polyhedral and

$$k \geq \min\left(m,\ |\{b_i \neq 0\}_{i=1}^m| + 1\right),$$

then

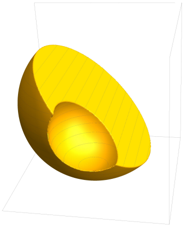
$$\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}} \quad \text{and} \quad \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}\,.$$

- $m = 1$
- A_is diagonal and $b_1 = b_2 = \cdots = b_m = 0$
- $A_i = \alpha_i I_n$ for all i and $n \geq m$

Example: Swiss cheese

- Minimizing distance to a piece of Swiss cheese

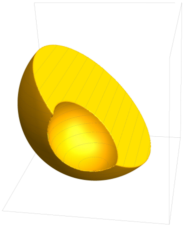
$$\inf_{x\in\mathbb{R}^n}\left\{\|x\|^2 : \begin{array}{l}\text{inside ball constraints}\\ \text{outside ball constraints}\\ \text{linear constraints}\end{array}\right\}$$



Example: Swiss cheese

- Minimizing distance to a piece of Swiss cheese

$$\inf_{x\in\mathbb{R}^n}\left\{\|x\|^2 : \begin{array}{l}\text{inside ball constraints}\\ \text{outside ball constraints}\\ \text{linear constraints}\end{array}\right\}$$

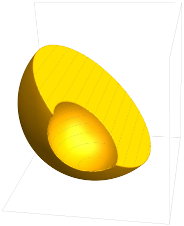


- inside ball $\mapsto I$, outside ball $\mapsto -I$, linear constraints $\mapsto 0$

Example: Swiss cheese

- Minimizing distance to a piece of Swiss cheese

$$\inf_{x\in\mathbb{R}^n} \left\{ \|x\|^2 : \begin{array}{l} \text{inside ball constraints} \\ \text{outside ball constraints} \\ \text{linear constraints} \end{array} \right\}$$



- inside ball $\mapsto I$, outside ball $\mapsto -I$, linear constraints $\mapsto 0$
- If nonempty and $n \geq m$, then the standard SDP relaxation is tight for this QCQP

Conclusion

- Analyzed

Conclusion

- Analyzed
  - the geometry of a dual object Γ

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\min\left(m, \lvert\{b_i \neq 0\}_{i=1}^m\rvert + 1\right)$	
SDP tightness	$\min\left(m, \lvert\{b_i \neq 0\}_{i=1}^m\rvert + 1\right)$	

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 2$
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 1$

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 2$
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 1$

- Results extend to equalities!

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 2$
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 1$

- Results extend to equalities!
- Future directions

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 2$
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 1$

- Results extend to equalities!
- Future directions
  - What if assumptions only approximately satisfied?

Conclusion

- Analyzed
  - the geometry of a dual object Γ
  - "amount of symmetry" k
- Assuming primal feasibility and dual strict feasibility, if $k \geq$

	Polyhedral Γ	General Γ
Convex hull result	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 2$
SDP tightness	$\operatorname{aff}\dim(\{b(\gamma) : \gamma \in \mathcal{F}\}) + 1$	$m + 1$

- Results extend to equalities!
- Future directions
  - What if assumptions only approximately satisfied?
  - Can this framework be used to recover other convex hull/exactness results?

Thank you. Questions?

Slides

cs.cmu.edu/~alw1

Full version

A. L. Wang and F. Kılınç-Karzan. "On the tightness of SDP relaxations of QCQPs". In: *arXiv preprint* arXiv:1911.09195 (2019)

References I

A. Beck. "Quadratic matrix programming". In: *SIAM Journal on Optimization* 17.4 (2007), pp. 1224–1238.

A. Beck, Y. Drori, and M. Teboulle. "A new semidefinite programming relaxation scheme for a class of quadratic matrix problems". In: *Operations Research Letters* 40.4 (2012), pp. 298–302.

A. Ben-Tal and D. den Hertog. "Hidden conic quadratic representation of some nonconvex quadratic optimization problems". In: *Mathematical Programming* 143 (2014), pp. 1–29.

S. Burer. "A gentle, geometric introduction to copositive optimization". In: *Mathematical Programming* 151 (2015), pp. 89–116.

References II

S. Burer and F. Kılınç-Karzan. "How to convexify the intersection of a second order cone and a nonconvex quadratic". In: *Mathematical Programming* 162 (2017), pp. 393–429.

S. Burer and Y. Ye. "Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs". In: *Mathematical Programming* 181 (2019), pp. 1–17.

A. L. Fradkov and V. A. Yakubovich. "The S-procedure and duality relations in nonconvex problems of quadratic programming". In: *Vestnik Leningrad Univ. Math.* 6 (1979), pp. 101–109.

M. Locatelli. "Exactness conditions for an SDP relaxation of the extended trust region problem". In: *Optimization Letters* 10.6 (2016), pp. 1141–1151.

References III

A. Santana and S. S. Dey. "The convex hull of a quadratic constraint over a polytope". In: *arXiv preprint* arXiv:1812.10160 (2018).

A. L. Wang and F. Kılınç-Karzan. "The Generalized Trust Region Subproblem: solution complexity and convex hull results". In: *arXiv preprint* arXiv:1907.08843 (2019).

B. Yang, K. Anstreicher, and S. Burer. "Quadratic programs with hollows". In: *Mathematical Programming* 170.2 (2018), pp. 541–553.

U. Yıldıran. "Convex hull of two quadratic constraints is an LMI set". In: *IMA Journal of Mathematical Control and Information* 26.4 (2009), pp. 417–450.