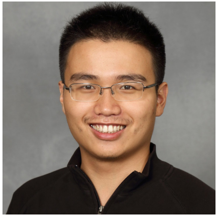
New first-order methods in modern/classical settings

Daniels School of Business Quantitative Methods Seminar

September 2023



Lijun Ding
UW-Madison

Ben Grimmer
Johns Hopkins

Kevin Shu
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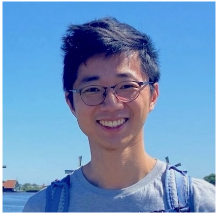

Alex L. Wang
Purdue University

Part 1

Sharp exact penalty formulations in signal recovery

Joint work with Lijun Ding

Outline

- Motivation: Sparse recovery and low-rank covariance estimation

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  - $\longrightarrow$ Abstract signal recovery problem

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  - $\longrightarrow$ Abstract signal recovery problem
- A new formulation of the abstract problem that is sharp
  - Better robustness guarantees, faster algorithms
- Numerical results

Motivation: Sparse recovery and covariance estimation

Sparse recovery setup

- **Recovery task**: Recover $x^\sharp \in \mathbb{R}^n$ from $A \in \mathbb{R}^{m\times n}, b = Ax^\sharp$

Related: Candes and Tao [2005], Recht et al. [2010], Candès et al. [2013]

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- Suppose A entrywise i.i.d. $N(0, 1/m^2)$

$$\left|\text{supp}(x^\sharp)\right| \le k \ll n \qquad m \asymp k\log(n)$$

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- **Conceptual approach:** $\min_{x\in\mathbb{R}^n} \left\{ |\mathrm{supp}(x)| \;:\; Ax = b \right\}$

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- **Conceptual approach:** $\min_{x \in \mathbb{R}^n} \left\{ |\mathrm{supp}(x)| \ : \ Ax = b \right\}$
- **Convex optimization approach**: In this regime, $x^\sharp$ is unique minimizer of

$$\min_{x \in \mathbb{R}^n} \left\{ \|x\|_1 \ : \ Ax = b \right\}$$

Related: Candes and Tao [2005], Recht et al. [2010], Candès et al. [2013]

Low-rank covariance estimation

- **Recovery task**: Recover $X^\sharp \in \mathbb{S}^n_+$ with $\mathrm{rank}(X^\sharp) \le k$ from $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, $b = \mathcal{A}(X^\sharp)$

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- Suppose $\mathcal{A}^*(e_i) = a_i a_i^\intercal$ where $a_i \sim N(0, I_n/m)$ and $m \asymp nk$

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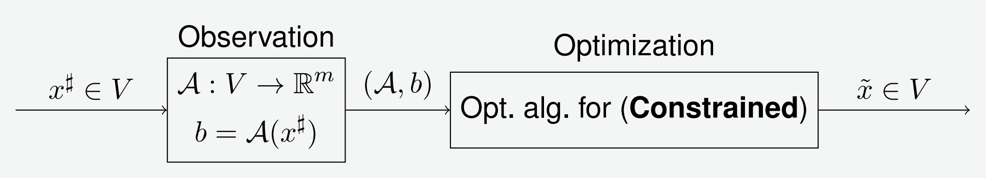
Abstract signal recovery problem and questions



(Constrained)

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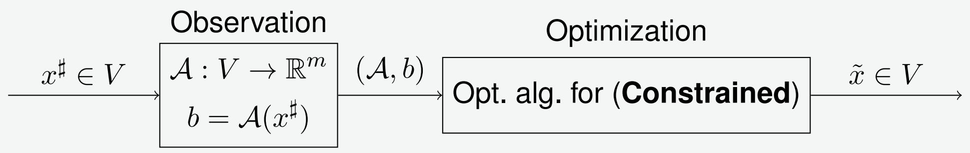

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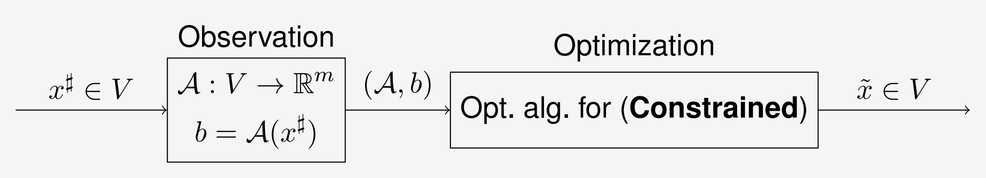

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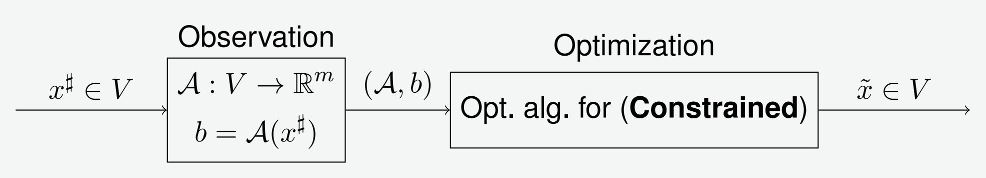

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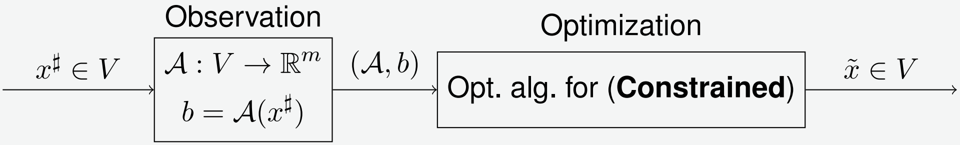

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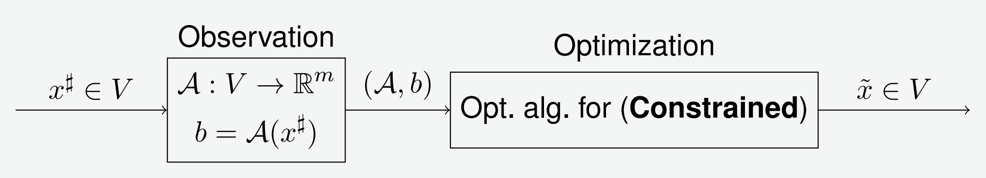

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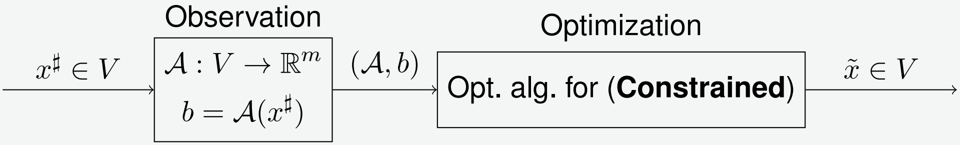

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  - Another convex problem?

A sharp penalty formulation

A penalty formulation

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Related: Beck and Teboulle [2009], Tibshirani [1996]

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- **(Constrained)** $$\min_{x \in V} \left\{ f(x) : \begin{array}{l} \mathcal{A}(x) = b \\ x \in K \end{array} \right\}$$
- Penalty formulation: let $r \asymp \sqrt{k}$ be a penalty parameter

$$F(x) := f(x) + r \left\| \mathcal{A}(x) - b \right\|_1 + 2 \operatorname{dist}_1(x, K)$$

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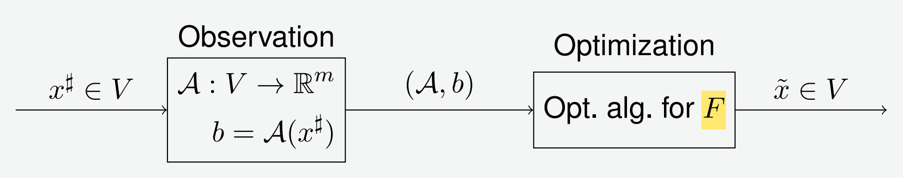

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Theorem (Structural)

F is μ-sharp in the ℓ_1 norm where μ is a function of “RIP constants of $\mathcal{A}$”

$$F(x) - F(x^\sharp) \geq \mu \left\|x - x^\sharp\right\|_1, \qquad \forall x \in V$$

and L-Lipschitz in the ℓ_1 norm with $L \asymp \sqrt{k}$

$$|F(x) - F(y)| \leq L \|x - y\|_1, \qquad \forall x, y.$$

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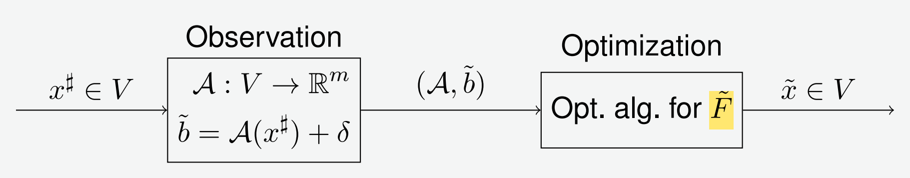
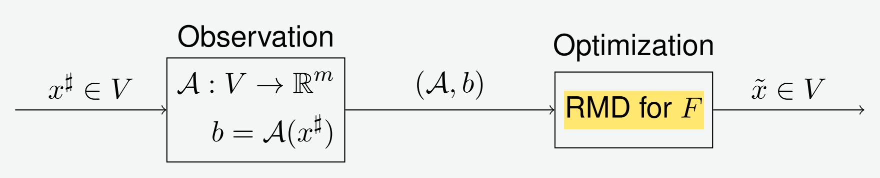


Corollary (Robustness)

Let $\tilde{x}$ be an ϵ minimizer of $\tilde{F}$.

- (to small noise) $\tilde{x}$ satisfies $\left\|\tilde{x} - x^\sharp\right\|_1 \lesssim \frac{\sqrt{k}}{\mu} \|\delta\|_1 + \frac{\epsilon}{\mu}$
- (to sparse noise) If $\frac{|\mathrm{supp}(\delta)|}{m} \lesssim 1/\sqrt{k}$, then $\left\|\tilde{x} - x^\sharp\right\|_1 \lesssim \frac{\epsilon}{\mu}$

Algorithms for minimizing F



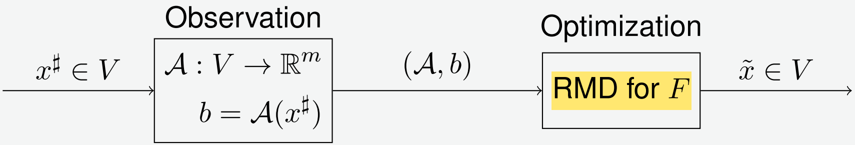
Corollary (Algorithms)

Restarted mirror descent (RMD) algorithm produces an ϵ-optimal solution to F in

$$O\left(\frac{k}{\mu^2} \log(n) \log(\epsilon^{-1})\right)$$

iterations of the mirror descent update.

Related: Polyak [1969], Roulet and d'Aspremont [2017], Yang and Lin [2018], Renegar and Grimmer [2022]



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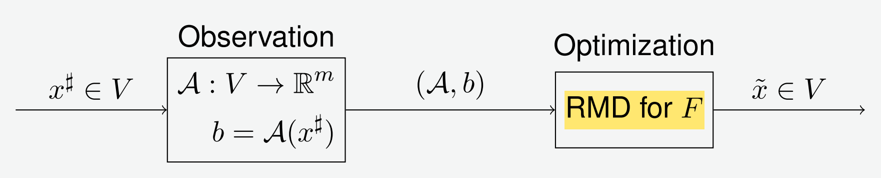

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- Requires μ
- If μ is not known, extra $\log(\epsilon^{-1})$ factor

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Algorithms for minimizing F

- Suppose we run MD from x_0 for t iterations with step size η and mirror map

$$h(x) \approx \frac{1}{2} \|x - x_0\|_1^2$$

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$$F(y) - F(x^\sharp) \leq \frac{L^2 \eta \ln n}{2} + \frac{D_h(x^\sharp || x_0)}{2\eta t}$$

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$$= L \left\|x^\sharp - x_0\right\|_1 \sqrt{\frac{\ln n}{2t}}$$

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- Applying sharpness $\longrightarrow$

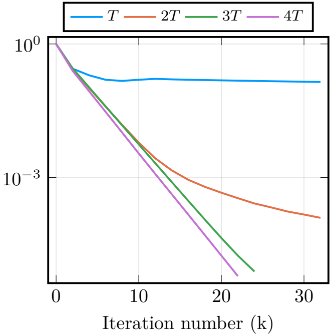
$$F(y) - F(x^\sharp) \leq \frac{1}{2} \left(F(x_0) - F(x^\sharp)\right)$$

after $\asymp \frac{L^2}{\mu^2} \ln n$ iterations

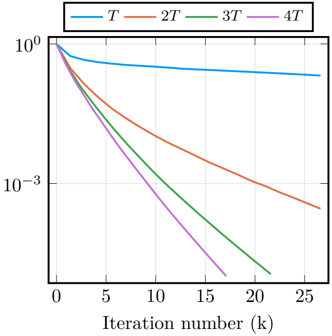
Related: Polyak [1969], Roulet and d'Aspremont [2017], Yang and Lin [2018], Renegar and Grimmer [2022]

Numerical experiments

- Let T be statistical threshold for sparse recovery, low-rank matrix sensing (covariance estimation without PSD constraint), and phase retrieval (covariance estimation with $k = 1$)



sparse recovery
$(n, k) = (10^4, 5)$



matrix sensing
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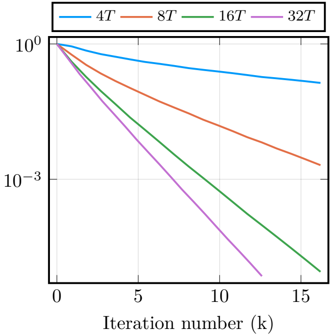
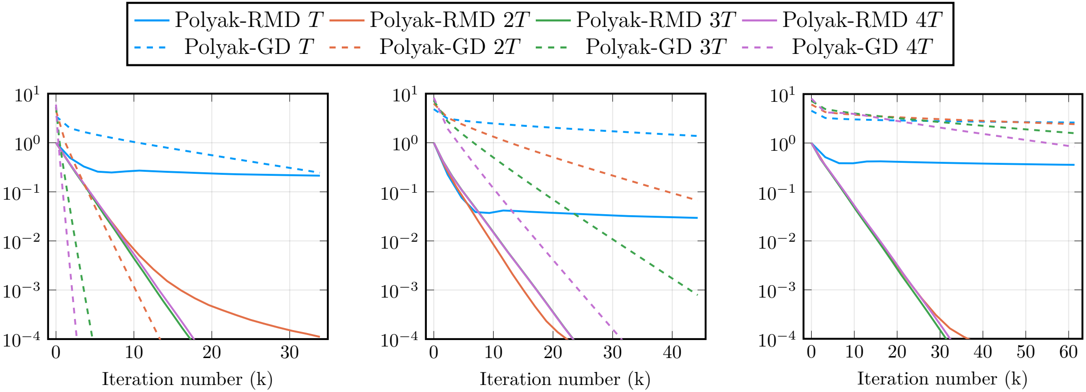


phase retrieval
$n = 100$

Restarted mirror descent vs. Polyak subgradient

- Polyak subgradient converges linearly on sharp Lipschitz functions in ℓ_2 norm



sparse recovery $(n, k) = (10^4, 5)$

sparse recovery $(n, k) = (10^5, 5)$

sparse recovery $(n, k) = (10^6, 5)$

Conclusion

- Abstract statistical signal recovery problem: sparse recovery, covariance estimation, matrix sensing, phase retrieval

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  - **Robustness**: to observation error and optimization error
  - **Algorithms**: Restarted Mirror Descent converges linearly

Questions?

Part 2

$O(1/T^{1.02449})$ Convergence of long-step gradient descent

Joint work with Benjamin Grimmer, Kevin Shu

Outline

- Preview of results (better guarantees for smooth convex minimization)

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- Why to expect this (history of prior works)

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- Conceptual contributions

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- Conceptual contributions
- Computer assisted design/proofs

Preview of results

Smooth convex optimization and gradient descent

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Smooth convex optimization and gradient descent

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Smooth convex optimization and gradient descent

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  - f is 1-smooth
  - f has minimizer $x^\star$
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Smooth convex optimization and gradient descent

- Want gradient descent-style algorithms for general convex functions f with
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- **Goal**: pick steplength sequence $(h_0, h_1, \dots)$ to maximize convergence rate

What we knew prior to 2021

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- Optimal rates for first-order methods: Accelerated gradient descent

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  Note: this is *not* a gradient descent-style algorithm

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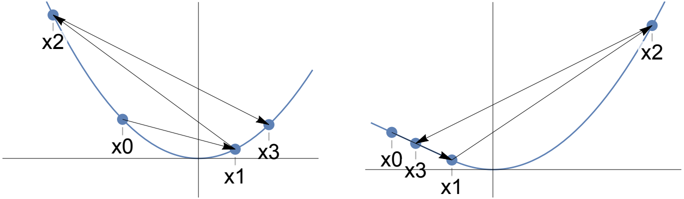

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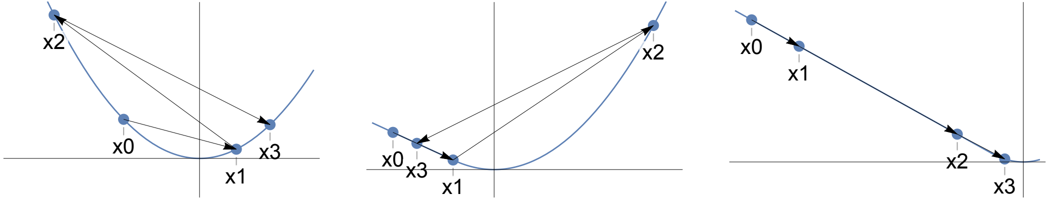

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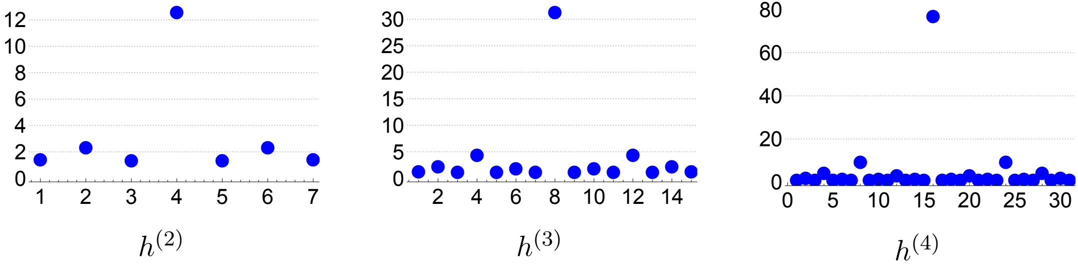

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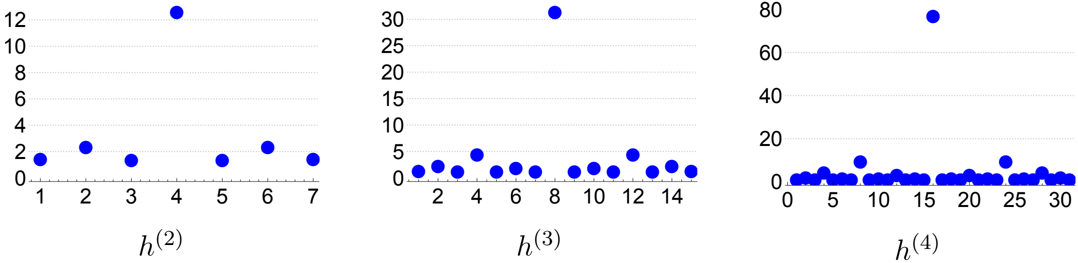

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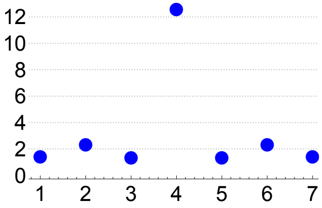

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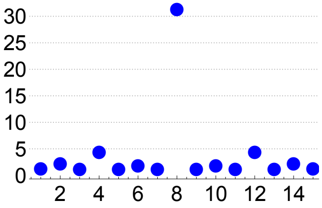

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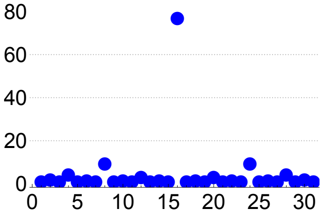
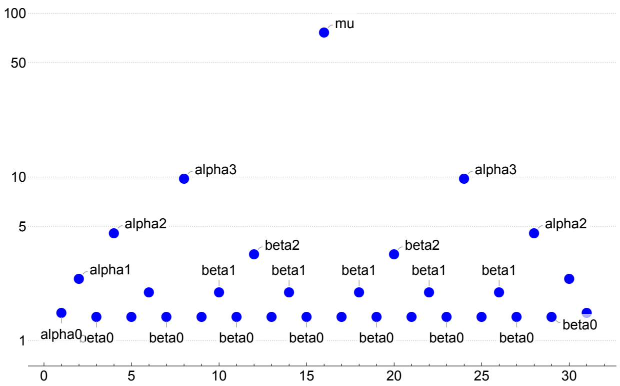


$h^{(4)}$

- Longer patterns have increasingly fast convergence rates
- $\mathrm{avg}(h^{(k)})$ is exponential in k

A closer look at $h^{(k)}$



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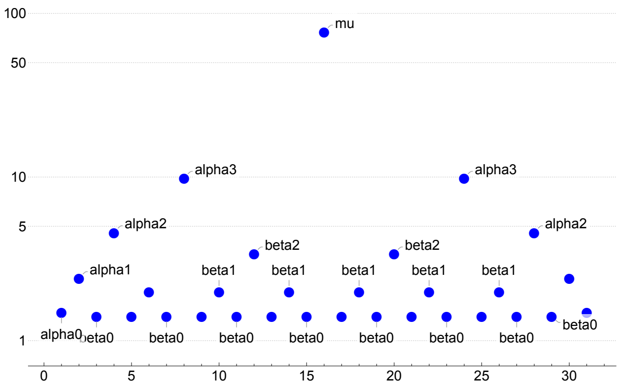

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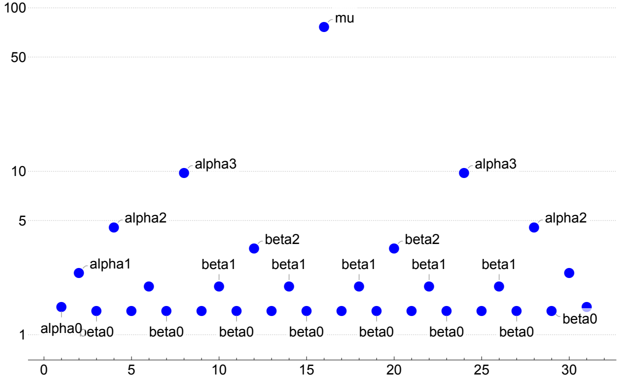

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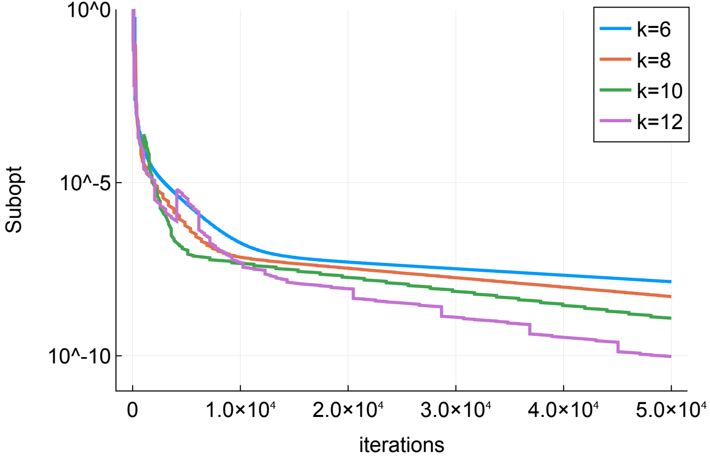
- α_i picked so that $\prod_{\text{stepsizes}}(\text{stepsize} - 1) = 1$

Numerical comparison of $h^{(k)}$



- $h^{(12)}$ has length 8191

Theorem

Suppose

$$h = \frac{1}{2}\left(\boxed{h^{(0)}, \ldots, h^{(0)}}, \boxed{h^{(1)}, \ldots, h^{(1)}}, \ldots, \boxed{h^{(k)}, \ldots, h^{(k)}}, \ldots\right)$$

where each $h^{(k)}$ is repeated $\approx c^k$ times. Then

$$\left(\min_{t \le T} f(x_t)\right) - f(x^\star) = O\left(\frac{1}{T^{1.02449}}\right)$$

Why should we expect this?

AKA some recent work in the area

The Performance Estimation Problem (PEP) 1/2

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Gradient descent with long steps

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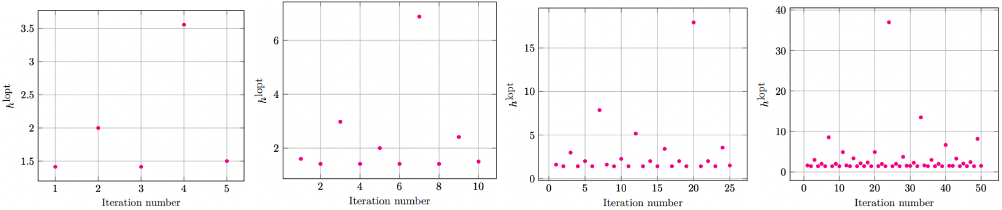

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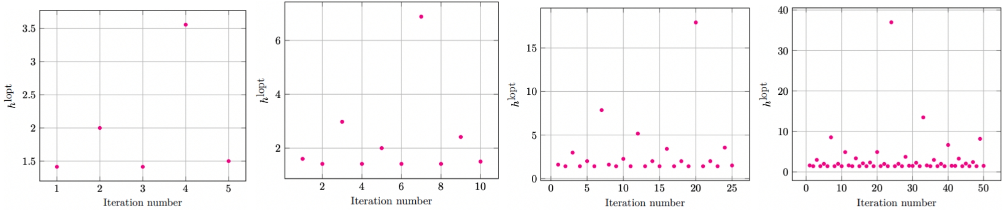

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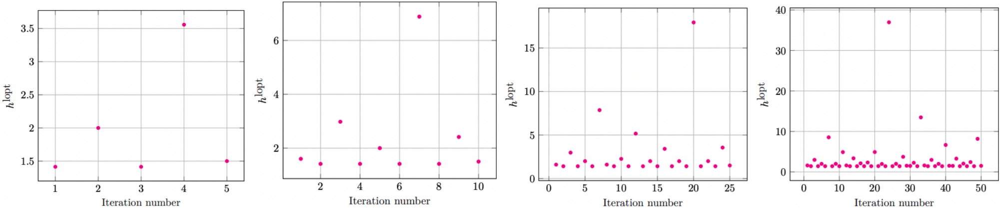

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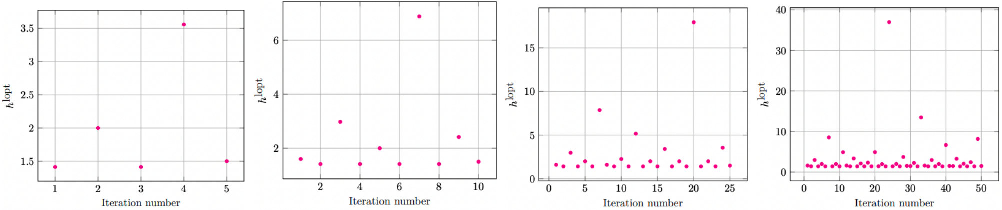

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- Our work: analytically construct solution for all δ, with $p_h(\delta)$ small

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Conceptual contributions

A suitable induction

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- Suppose $\delta > 0$ small and consider

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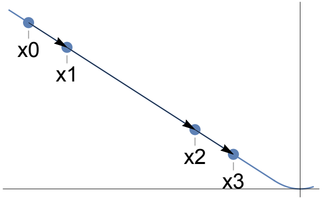

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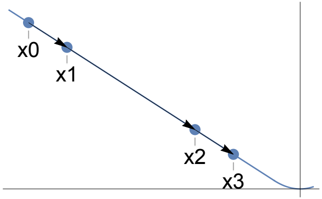

- Then, $f(x_0) = \delta$ and $f(x_T) = \delta - \delta^2 \sum h_i$

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$$f(x) = \begin{cases} -\delta x - \delta^2/2 & \text{if } x \leq -\delta \\ \frac{1}{2}x^2 & \text{if } x \geq -\delta \end{cases}$$

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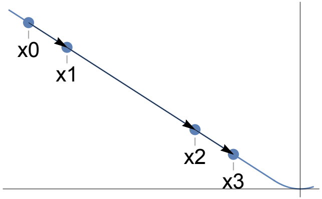

- Then, $f(x_0) = \delta$ and $f(x_T) = \delta - \delta^2 \sum h_i$
- Thus, for $\delta > 0$ small, $p_h(\delta) \geq \delta - \delta^2 \sum h_i$

Straightforward blocks

- We say a steplength *block* $h = (h_0, h_1, \ldots, h_{T-1})$ is Δ-straightforward if

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- We show that $\text{avg}(h^{(k)})$ is exponentially large in k, $\Delta^{(k)}$ is $\geq$ exponentially small $\longrightarrow$ accelerated convergence rates

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- This becomes a nonlinear SDP but can be "reformulated" into a regular SDP if we consider "limiting behavior as $\Delta \to 0$", at which point we can attempt to certify Δ-straightforwardness computationally

Computer assisted design/proofs

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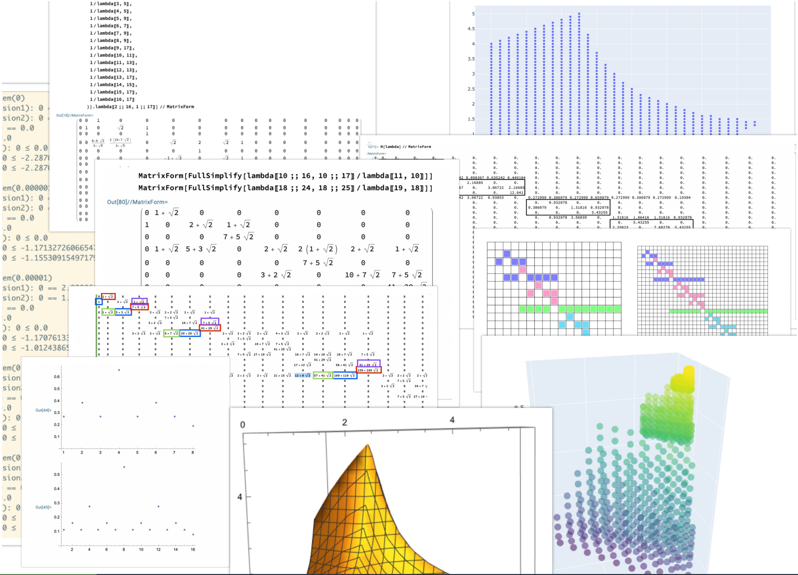
Pictures of process



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Questions?

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