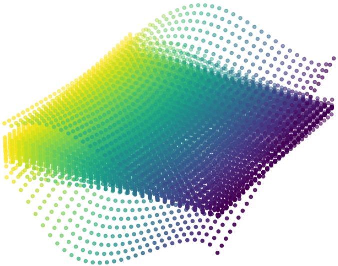
New notions of simultaneous diagonalizability of quadratic forms with applications to QCQPs

Alex L. Wang, MOPTA, Aug. 21



Joint work with Rujun Jiang, Fudan University

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- SDP relaxations more tractable[1]
- Better understanding of exactness of relaxations[2]
- Black-box global solvers seem to perform better

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- $\{A_i\}$ is SDC $\iff$ $\exists \{\ell_1, \dots, \ell_n\} \subseteq \mathbb{R}^n$: basis

$$A_i = \sum_j \mu_j^{(i)} \ell_j \ell_j^\intercal, \quad \forall i$$

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$\{A_i\} \subseteq \mathbb{S}^n$ is d-Restricted SDC if there exists $\{\overline{A}_i\} \subseteq \mathbb{S}^{n+d}$ SDC

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- Tools: canonical form for pairs of symmetric matrices[3]

[3] [Uhlig 76], [Lancaster, Rodman 05]

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- $$\begin{aligned} \inf_{x\in\mathbb{R}^n} \quad & x^\intercal A_1 x \\ \text{s.t.} \quad & x^\intercal A_2 x \leq 0 \\ & Lx \leq 1 \end{aligned}$$

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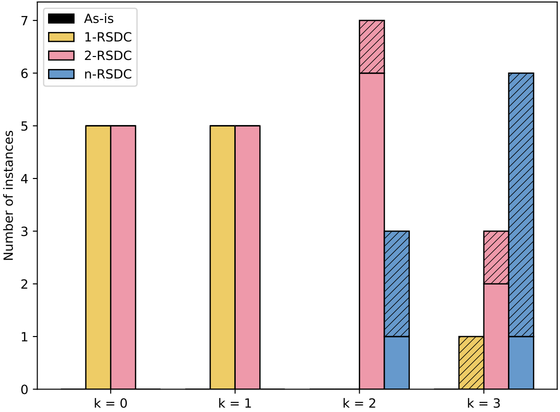

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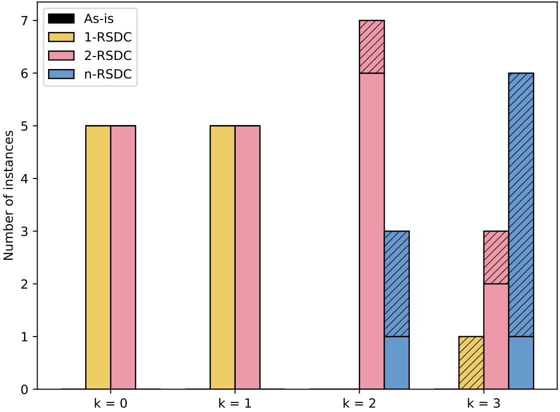

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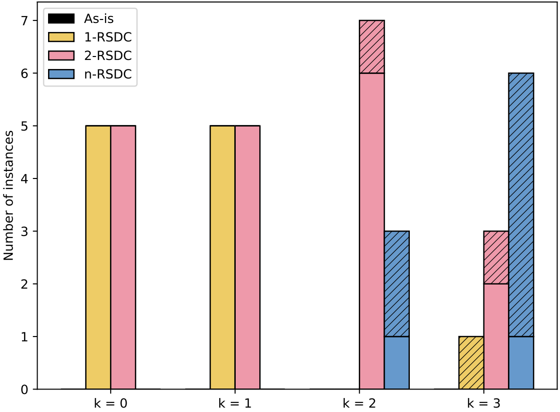

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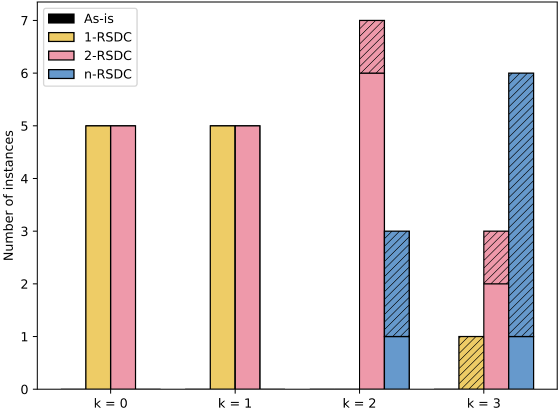

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  - $k = 3$: 1-RSDC ($\sim 10^3$), 2-RSDC ($\sim 10^2$), n-RSDC (1)

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- Thank you. Questions?

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