MGMT 690 - Pset 1

Spring 2024

Instructions:

- This pset is due on Sunday, March 24 at 11:59pm.
- Completed psets should be submitted to Gradescope.
- **Exercises** are for your own review only. They do not need to be submitted and will not be graded.
- **Complete all problems 1–3 and *one of either* 4 or 5.**

Exercises

1. Let V be a Euclidean space and let
$$\|v\| := \sqrt{\langle v, v\rangle}.$$
Prove that this is a norm.

2. Let $p \in > 0$. For $x \in \mathbb{R}^n$, define
$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$
Prove that this is *not* a norm for $p \in (0,1)$ and $n \geq 2$.

3. Prove that the affine image of a convex set is a convex set.

4. Let $C \subseteq \mathbb{R}^n$ be a convex set. Let $x \in \mathrm{rint}(C)$ and $y \in \mathrm{cl}(C)$. Prove that for all $\theta \in [0,1)$, that $(1-\theta)x + \theta y \in \mathrm{rint}(C)$.

Problems

1. [25 pts] Given $A \in \mathbb{S}^n$ and $B \in \mathbb{S}^m$, the Kronecker product $A \otimes B$ is the $\mathbb{S}^{mn}$ matrix given in block form as
$$A \otimes B = \begin{pmatrix} A_{1,1}B & \dots & A_{1,n}B \\ \vdots & \ddots & \vdots \\ A_{n,1}B & \dots & A_{n,n}B \end{pmatrix}$$
Suppose $A \in \mathbb{S}^n_+$ and $B \in \mathbb{S}^m_+$. Show that $A \otimes B \succeq 0$.

2. [25 pts] Given a symmetric matrix $A \in \mathbb{S}^n$, let $\operatorname{Inertia}(A) \coloneqq (n_-, n_0, n_+)$ denote the number of negative eigenvalues, number of zero eigenvalues, and number of positive eigenvalues of A. Prove that for any invertible $P \in \mathbb{R}^{n\times n}$, that

$$\operatorname{Inertia}(A) = \operatorname{Inertia}(P^\intercal A P).$$

3. [25 pts] Prove that

   (a) [5pts] the nonnegative orthant is self-dual,

   (b) [10pts] the second-order cone is self-dual, and

   (c) [10pts] the semidefinite cone is self-dual.

4. [25 pts] In sparse recovery, the goal is to recover a sparse vector $x^\star \in \mathbb{R}^n$ given linear measurements $(A, b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ where $b = Ax^\star$. A convex-optimization approach to this problem is to output the optimizer of

$$\min_{x\in\mathbb{R}^n} \left\{\|x\|_1 : Ax = b\right\}.$$

   This problem gives a necessary and sufficient condition for when this convex-optimization approach correctly recovers $x^\star$.

   We say that a vector is k-sparse if it has at most k nonzero entries. Given a subset $S \subseteq [n]$ and a vector $x \in \mathbb{R}^n$, let x_S denote the restriction of x onto the set S. Let S^c denote the complement of S. For a vector $x \in \mathbb{R}^n$, let $\operatorname{sign}(x)$ denote the $\{-1, 0, 1\}$-valued vector giving the individual signs of the coordinates of x.

   (a) [10pts] The *descent cone* of a convex-optimization problem at a feasible solution $\bar{x}$ is defined as

$$\left\{ \delta \in \mathbb{R}^n : \begin{array}{l} \forall \epsilon > 0 \text{ small enough :} \\ \bar{x} + \epsilon\delta \text{ is feasible} \\ \text{obj. value at } \bar{x} + \epsilon\delta \leq \text{obj. value at } \bar{x} \end{array} \right\}$$

   Show that for this problem, the descent cone at the optimal solution $x^\star$ is

$$\left\{ \delta \in \mathbb{R}^n : \begin{array}{l} \delta \in \ker(A) \\ \langle \operatorname{sign}(x^\star), \delta_{S^\star}\rangle + \left\|\delta_{(S^\star)^C}\right\|_1 \leq 0 \end{array} \right\}$$

   where $S^\star$ is the support of $x^\star$.

   (b) [10pts] The matrix A is said to satisfy the *nullspace property at order* k if for all sets $S \subseteq [n]$ with $|S| \leq k$ and for all $\delta \in \ker(A) \setminus \{0\}$, we have

$$\|\delta_S\|_1 < \|\delta_{S^c}\|_1.$$

   Show that the descent cone at $x^\star$ is trivial, i.e., equal to $\{0\}$, if A satisfies the nullspace property at order k and $x^\star$ is k-sparse.

(c) [5pts] Show that if A does not satisfy the nullspace property at order k, then there exists a k-sparse $x^\star$ for which the convex-optimization approach may fail to recover $x^\star$. That is, for which the descent cone at $x^\star$ is nontrivial.

5. [25 pts] Given a permutation σ of $[n]$, we can associate σ with the $n \times n$ permutation matrix

$$(X^\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{else} \end{cases}.$$

Let $\mathcal{P}_n$ denote the set of $n!$ permutation matrices of size $n \times n$. Prove that $\mathrm{conv}(\mathcal{P}_n) = \mathrm{DS}_n$, the set of doubly stochastic matrices:

$$\mathrm{DS}_n := \left\{ X \in \mathbb{R}^{n \times n} : \begin{array}{c} X \geq 0 \\ X^\intercal 1_n = 1_n \\ X 1_n = 1_n \end{array} \right\}.$$

Hint: Use Hall's marriage theorem to prove that the support of any doubly stochastic matrix contains a permutation matrix.