MGMT 690 - Pset 2

Spring 2024

Instructions:

- This pset is due on Sunday, April 7 at 11:59pm.
- Completed psets should be submitted to Gradescope.
- **Exercises** are for your own review only. They do not need to be submitted and will not be graded.
- **Complete all problems 1–4.**

Exercises

- Suppose $f_0, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ are SOCR functions and $\mathcal{X}_1, \ldots, \mathcal{X}_m \subseteq \mathbb{R}^n$ are SOCR set. Rewrite the following optimization problem as an SOCP

$$\inf_{x \in \mathbb{R}^n} \left\{ f_0(x) : \begin{array}{l} f_i(x) \leq 0, \, \forall i \in [k] \\ x \in \mathcal{X}_i, \, \forall i \in [m] \end{array} \right\}.$$

- Recall that $S_k : \mathbb{S}^n \to \mathbb{R}$ is the function where $S_k(X)$ is the sum of the k largest eigenvalues of X. Prove that

$$S_k(X) = \max_{Y \in \mathbb{S}^n} \left\{ \langle X, Y \rangle : \begin{array}{l} 0 \preceq Y \preceq I \\ \operatorname{tr}(Y) = k \end{array} \right\}.$$

  Use this fact to deduce that S_k is convex. What is the dual to this problem? Compare the dual to the construction in Example 16 of the Lecture notes.

Problems

1. [15 pts] The following example shows that strong duality may fail in general for conic programs without further assumptions. Consider the following SDP.

$$\inf_{X \in \mathbb{S}^2} \left\{ 2X_{1,2} : \begin{array}{l} X_{1,1} = 0 \\ X \succeq 0 \end{array} \right\}$$

   Write its dual and compute the optimal value for both the primal and dual.

2. [15 pts] This problem derives a dual description of the *Wasserstein distance* for discrete probability distributions.

   Fix a discrete metric space $\mathcal{X} = \{x_1, \ldots, x_n\}$. Let $D \in \mathbb{R}^{n\times n}$ denote the matrix where $D_{i,j}$ is the distance between x_i and x_j.

   Let P be a probability distribution on $\mathcal{X}$ defined by $P = (p_1, \ldots, p_n)$. Similarly, let $Q = (q_1, \ldots, q_n)$ be a probability distribution on $\mathcal{X}$.

   The *Wasserstein distance* between P and Q is defined as follows: We can think of P as placing some amount of "earth/dirt" at each of the n points in $\mathcal{X}$. We want to move this earth as efficiently as possible to transform P into Q. That is, we require a transportation schedule, called a *coupling*, that tells us how much earth to move from x_i to x_j. Formally, the matrix $\Gamma \in \mathbb{R}^{n\times n}$ is a coupling if

$$\begin{aligned} \sum_{j=1}^n \Gamma_{i,j} &= p_i, \qquad \forall i \in [n] \\ \sum_{i=1}^n \Gamma_{i,j} &= q_j, \qquad \forall j \in [n] \\ \Gamma_{i,j} &\geq 0, \qquad \forall i, j. \end{aligned}$$

   The cost of a coupling is given by $\langle \Gamma, D\rangle$, i.e., it is the linear cost function where moving one unit of mass from x_i to x_j costs $D_{i,j}$.

   - [5 pts] Write the Wasserstein distance as the optimum value of a minimization LP. We will refer to this as the primal LP.
   - [5 pts] Derive the dual of this LP.
   - [5 pts] Explain why both primal and dual are solvable. Explain what complementary slackness means for this primal-dual pair.

3. [10 pts] Show that the following branch of the hyperbola is a SOCR set.

$$\left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{c} xy \geq 1 \\ x, y \geq 0 \end{array} \right\}$$

4. [30 pts] Consider an optimization problem of the form

$$\inf_{x\in\mathbb{R}^n} \left\{ f(x) : \, g_i(x) \leq 0, \, \forall i \in [m] \right\}.$$

   We make no assumptions on whether f or $g_1, \ldots, g_m$ is convex. Define

$$\mathcal{I} := \left\{ \begin{pmatrix} f(x) \\ g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix} : \, x \in \mathbb{R}^n \right\} + \mathbb{R}_+^{1+m}.$$

- [10 pts] Show that if f and $g_1, \dots, g_m$ are convex functions, then $\mathcal{I}$ is a convex set.
- [20 pts] We now will only assume that $\mathcal{I}$ is a convex set (while $f, g_1, \dots, g_m$ may not necessarily be convex).

  Adapt the proof of strong conic duality to show that if $\mathcal{I}$ is convex and there exists $\bar{x}$ so that $g_i(\bar{x}) < 0$ for all $i \in [m]$, then

$$\begin{aligned} &\inf_{x\in\mathbb{R}^n} \{f(x) : \, g_i(x) \leq 0, \, \forall i \in [m]\} \\ &= \sup_{u\in\mathbb{R}, \, \lambda\in\mathbb{R}^m} \left\{ u : \begin{array}{l} \lambda \geq 0 \\ f(x) + \sum_{i=1}^m \lambda_i g_i(x) \geq u, \, \forall x \in \mathbb{R}^n \end{array} \right\} \end{aligned}$$

  where the dual problem is solvable (i.e., the supremum is achieved).

This statement is known as hidden convexity and allows us to extend convex optimization theory to some very special nonconvex optimization problems where $\mathcal{I}$ is convex despite f, g_i being possibly nonconvex.

5. [30 pts] Suppose K is a proper cone and consider the primal and dual conic problems:

$$\inf_{x\in\mathbb{R}^n} \left\{ c^\intercal x : \begin{array}{l} Ax - a \in K \\ Bx - b = 0 \end{array} \right\} \geq \sup_{y\in\mathbb{R}^m, \, z\in\mathbb{R}^k} \left\{ \langle a, y\rangle + \langle b, z\rangle : \begin{array}{l} A^\intercal y + B^\intercal z = c \\ y \in K_* \end{array} \right\}.$$

Furthermore, assume that the primal problem is feasible and that:

$$\ker\left(\begin{pmatrix} c^\intercal \\ A \\ B \end{pmatrix}\right) = \{0\}.$$

Prove that the primal problem has bounded sublevel sets, i.e.,

$$\forall t \in \mathbb{R}, \ \text{ the set } \left\{ x \in \mathbb{R}^n : \begin{array}{l} c^\intercal x \leq t \\ Ax - a \in K \\ Bx - b = 0 \end{array} \right\} \text{ is bounded}$$

if and only if the dual problem is strictly feasible.

Hint: in the only if direction, consider the set

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} c^\intercal x \leq 0 \\ Ax \in K \\ Bx = 0 \end{array} \right\} = \{0\}.$$

You must justify why this set needs to be $\{0\}$. Now, take the dual cone of either side of this equation. You may use the fact that the relative interior of an affine image of a convex set is the affine image of the relative interior of the convex set.