Proof of the Quadratic Formula

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While it is not necessary to be able to prove anything in this course, an algebraic reasoning behind a formula that we use often certainly couldn't hurt. say we have some quadratic with arbitrary coefficients a, b, and c and we are interested in the zeroes:

$$ax^2 + bx + c = 0$$

We proceed by putting our quadratic equation into standard form by completing the square.

$$\begin{aligned}
&ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c
\end{aligned}$$

Now that the equation is in standard form let us find its zeroes explicitly.

$$\begin{aligned}
&a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0 \\
&a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \\
&\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
&x = \frac{-b}{2a} + \frac{\pm\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

We arrive at the well known formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Naturally one might ask: if there is a formula to find the roots of a general quadratic, does there exist such a formula for a general cubic, quartic?

The answer is yes, there do exist formulas like this for both general cubics and quartics, however they are much, much, more complicated and are impossible to memorize, unlike the the quadratic formula. While these types of formulas do exist for cubics and quartics as well, it is a well-known fact in mathematics that there exists no such formula for a polynomial of degree 5 or higher.