Technical Appendix (not for publication)

Derivation of Expressions in the Bias Result in Section 2.1

Taking the derivative of the score function in (2.3), we have the Hessian matrix

$$\boldsymbol{H}_n(\boldsymbol{\theta}) = -\begin{bmatrix} \frac{\boldsymbol{X}_n'\boldsymbol{X}_n}{n\sigma^2} & * & * \\ \frac{\boldsymbol{f}_n'\boldsymbol{X}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{X}_n}{n\sigma^2} & \frac{\boldsymbol{f}_n'\boldsymbol{f}_n+2\boldsymbol{f}_n'\boldsymbol{G}_n\boldsymbol{v}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{G}_n\boldsymbol{v}_n+\sigma^2\mathrm{tr}[\boldsymbol{G}_n^2(\lambda)]}{n\sigma^2} & * \\ \frac{\boldsymbol{v}_n'(\boldsymbol{\delta})\boldsymbol{X}_n}{n\sigma^4} & \frac{\boldsymbol{v}_n'(\boldsymbol{\delta})\boldsymbol{f}_n+\boldsymbol{v}_n'(\boldsymbol{\delta})\boldsymbol{G}_n\boldsymbol{v}_n}{n\sigma^4} & \frac{\boldsymbol{v}_n'(\boldsymbol{\delta})\boldsymbol{v}_n(\boldsymbol{\delta})}{n\sigma^6}-\frac{1}{2\sigma^4} \end{bmatrix}.$$

The $(k+2)\times(k+2)^2$ matrix $\partial\boldsymbol{H}_n(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ is defined recursively as follows:

$$\begin{bmatrix} \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'\partial\sigma^2} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\lambda\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\lambda\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\lambda\partial\sigma^2} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\sigma^2\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\sigma^2\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\sigma^2\partial\sigma^2} \\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\boldsymbol{\beta}'\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\boldsymbol{\beta}'\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\boldsymbol{\beta}'\partial\sigma^2} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\lambda\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\lambda\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\lambda\partial\sigma^2} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\sigma^2\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\sigma^2\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\sigma^2\partial\sigma^2} \\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\boldsymbol{\beta}'\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\boldsymbol{\beta}'\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\boldsymbol{\beta}'\partial\sigma^2} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\lambda\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\lambda\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\lambda\partial\sigma^2} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\sigma^2\partial\boldsymbol{\beta}'} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\sigma^2\partial\lambda} & \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\sigma^2\partial\sigma^2} \end{bmatrix}$$

where the nonzero (unique) blocks/elements are

$$\begin{aligned} \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'\partial\sigma^2} &= \frac{\boldsymbol{X}_n'\boldsymbol{X}_n}{n\sigma^4},\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\lambda\partial\sigma^2} = \frac{\boldsymbol{X}_n'\boldsymbol{f}_n+\boldsymbol{X}_n'\boldsymbol{G}_n\boldsymbol{v}_n}{n\sigma^4},\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\sigma^2\partial\sigma^2} = \frac{2\boldsymbol{X}_n'\boldsymbol{v}_n(\boldsymbol{\delta})}{n\sigma^6}, \\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\boldsymbol{\beta}'\partial\sigma^2} &= \frac{\boldsymbol{f}_n'\boldsymbol{X}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{X}_n}{n\sigma^4},\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\lambda\partial\lambda} = -\frac{2\mathrm{tr}[\boldsymbol{G}_n^3(\lambda)]}{n}, \\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\lambda\partial\sigma^2} &= \frac{\boldsymbol{f}_n'\boldsymbol{f}_n+2\boldsymbol{f}_n'\boldsymbol{G}_n\boldsymbol{v}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{G}_n\boldsymbol{v}_n}{n\sigma^4},\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\lambda\partial\sigma^2\partial\sigma^2} = \frac{2\boldsymbol{f}_n'\boldsymbol{v}_n(\boldsymbol{\delta})+2\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{v}_n(\boldsymbol{\delta})}{n\sigma^6}, \\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\boldsymbol{\beta}'\partial\boldsymbol{\beta}'} &= \frac{[\mathrm{vec}(\boldsymbol{X}_n'\boldsymbol{X}_n)]'}{n\sigma^4},\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\boldsymbol{\beta}'\partial\sigma^2} = \frac{2\boldsymbol{v}_n(\boldsymbol{\delta})'\boldsymbol{X}_n}{n\sigma^6},\ \frac{\partial^3\mathcal{L}_n(\boldsymbol{\theta})}{\partial\sigma^2\partial\sigma^2\partial\sigma^2} = \frac{3\boldsymbol{v}_n(\boldsymbol{\delta})'\boldsymbol{v}_n(\boldsymbol{\delta})-n\sigma^2}{n\sigma^6}. \end{aligned}$$

Evaluating $\partial\boldsymbol{H}_n(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ at $\boldsymbol{\theta}=\boldsymbol{\theta}_0$ and taking expectation, we have the express for $\boldsymbol{D}_n$ as given in (2.9).

By substitution, we write

$$\mathrm{E}(\boldsymbol{H}_n\otimes\boldsymbol{\psi}_n') = \begin{bmatrix} \mathbf{O}_{k\times(k^2+2k)} & -\frac{1}{n\sigma_0^2}\mathrm{E}[(\boldsymbol{X}_n'\boldsymbol{f}_n+\boldsymbol{X}_n'\boldsymbol{G}_n\boldsymbol{v}_n)\otimes\boldsymbol{\psi}_n'] & -\frac{1}{n\sigma_0^4}\mathrm{E}[(\boldsymbol{X}_n'\boldsymbol{v}_n)\otimes\boldsymbol{\psi}_n'] \\ -\frac{1}{n\sigma_0^2}\mathrm{E}[(\boldsymbol{f}_n'\boldsymbol{X}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{X}_n)\otimes\boldsymbol{\psi}_n'] & \mathrm{E}(\boldsymbol{H}_{n,(k+1,k+1)}\boldsymbol{\psi}_n') & \mathrm{E}(\boldsymbol{H}_{n,(k+1,k+2)}\boldsymbol{\psi}_n') \\ -\frac{1}{n\sigma_0^4}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{X}_n)\otimes\boldsymbol{\psi}_n'] & \mathrm{E}(\boldsymbol{H}_{n,(k+2,k+1)}\boldsymbol{\psi}_n') & \mathrm{E}(\boldsymbol{H}_{n,(k+2,k+2)}\boldsymbol{\psi}_n') \end{bmatrix},$$

and

$$\mathrm{E}(\boldsymbol{\psi}_n\boldsymbol{\psi}_n') = \frac{1}{n\sigma_0^2}\begin{bmatrix} \mathrm{E}(\boldsymbol{X}_n'\boldsymbol{v}_n\boldsymbol{\psi}_n') \\ \mathrm{E}[(\boldsymbol{f}_n'\boldsymbol{v}_n)\boldsymbol{\psi}_n'] + \mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{\psi}_n'] \\ \frac{1}{2\sigma_0^2}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{v}_n)\boldsymbol{\psi}_n'] \end{bmatrix}.$$

Note that for any column vectors $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$, $\boldsymbol{a}_n'\otimes\boldsymbol{b}_n' = (\mathrm{vec}[(\boldsymbol{a}_n\otimes\boldsymbol{b}_n')'])'$ and $\boldsymbol{a}_n\otimes\boldsymbol{b}_n' = \boldsymbol{a}_n\boldsymbol{b}_n'$. So $(\boldsymbol{f}_n'\boldsymbol{X}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{X}_n)\otimes\boldsymbol{\psi}_n' = (\mathrm{vec}\,[((\boldsymbol{X}_n'\boldsymbol{f}_n+\boldsymbol{X}_n'\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{\psi}_n')'])'$ and $(\boldsymbol{v}_n'\boldsymbol{X}_n)\otimes\boldsymbol{\psi}_n' = (\mathrm{vec}[((\boldsymbol{X}_n'\boldsymbol{v}_n)\boldsymbol{\psi}_n')'])'$. Also, $\mathrm{E}[(\boldsymbol{X}_n'\boldsymbol{f}_n+\boldsymbol{X}_n'\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{\psi}_n'] = \mathrm{E}(\boldsymbol{X}_n'\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{\psi}_n')$ since $\mathrm{E}(\boldsymbol{\psi}_n)=\mathbf{0}_p$. Then what we need are the following expectations:

$$\mathrm{E}(\boldsymbol{X}_n'\boldsymbol{A}_n\boldsymbol{v}_n\boldsymbol{\psi}_n') = \frac{1}{n\sigma_0^2}\begin{bmatrix} \boldsymbol{X}_n'\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n')\boldsymbol{X}_n & \boldsymbol{X}_n'\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n')\boldsymbol{f}_n+\boldsymbol{X}_n'\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{v}_n) & \frac{1}{2\sigma_0^2}\boldsymbol{X}_n'\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{v}_n) \end{bmatrix},$$

for $\boldsymbol{A}_n = \boldsymbol{G}_n$ and $\boldsymbol{I}_n$, and

$$\begin{aligned}
\mathrm{E}(\boldsymbol{H}_{n,(k+1,k+1)}\boldsymbol{\psi}'_n) &= -\frac{2}{n\sigma_0^2}\mathrm{E}[(\boldsymbol{f}'_n\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n] - \frac{1}{n\sigma_0^2}\mathrm{E}[(\boldsymbol{v}'_n\boldsymbol{G}'_n\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n], \\
\mathrm{E}(\boldsymbol{H}_{n,(k+1,k+2)}\boldsymbol{\psi}'_n) &= -\frac{1}{n\sigma_0^4}\mathrm{E}[(\boldsymbol{f}'_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n] - \frac{1}{n\sigma_0^4}\mathrm{E}[(\boldsymbol{v}'_n\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n], \\
\mathrm{E}(\boldsymbol{H}_{n,(k+2,k+2)}\boldsymbol{\psi}'_n) &= -\frac{1}{n\sigma_0^6}\mathrm{E}[(\boldsymbol{v}'_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n].
\end{aligned}$$

In the above expressions,

$$\mathrm{E}[(\boldsymbol{f}'_n\boldsymbol{A}_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n] = \frac{1}{n\sigma_0^2}\left[\begin{array}{ccc} \boldsymbol{f}'_n\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}'_n)\boldsymbol{X}_n & \boldsymbol{f}'_n\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}'_n)\boldsymbol{f}_n + \boldsymbol{f}'_n\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}'_n\boldsymbol{G}'_n\boldsymbol{v}_n) & \frac{1}{2\sigma_0^2}\boldsymbol{f}'_n\boldsymbol{A}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}'_n\boldsymbol{v}_n) \end{array}\right],$$

for $\boldsymbol{A}_n = \boldsymbol{G}_n$ and $\boldsymbol{I}_n$, and

$$\mathrm{E}[(\boldsymbol{v}'_n\boldsymbol{A}_n\boldsymbol{v}_n)\boldsymbol{\psi}'_n]' = \frac{1}{n\sigma_0^2}\left[\begin{array}{c} \boldsymbol{X}'_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}'_n\boldsymbol{A}'_n\boldsymbol{v}_n) \\ \boldsymbol{f}'_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}'_n\boldsymbol{A}'_n\boldsymbol{v}_n) + \mathrm{E}(\boldsymbol{v}'_n\boldsymbol{A}_n\boldsymbol{v}_n\boldsymbol{v}'_n\boldsymbol{G}_n\boldsymbol{v}_n) - \sigma_0^2\mathrm{tr}(\boldsymbol{G}_n)\mathrm{E}(\boldsymbol{v}'_n\boldsymbol{A}_n\boldsymbol{v}_n) \\ \frac{1}{2\sigma_0^2}\mathrm{E}(\boldsymbol{v}'_n\boldsymbol{v}_n\boldsymbol{v}'_n\boldsymbol{A}_n\boldsymbol{v}_n) - \frac{n}{2}\mathrm{E}(\boldsymbol{v}'_n\boldsymbol{A}_n\boldsymbol{v}_n) \end{array}\right],$$

for $\boldsymbol{A}_n = \boldsymbol{G}_n$, $\boldsymbol{G}'_n\boldsymbol{G}_n$, and $\boldsymbol{I}_n$. All the expectations involved boil down to the expectations of quadratic and linear forms in the nonnormal vector $\boldsymbol{v}_n$, up to order 2, which can be found in Bao and Ullah (2010, "Expectation of quadratic forms in normal and nonnormal variables with applications", *Journal of Statistical Planning and Inference*, 140). Upon substitution and simplification, we have (2.10) and (2.11).

Order $O(h_n/n)$ Bias of $\hat{\lambda}_n$ under Divergent h_n

Here we outline the steps in evaluation of the approximate bias of the QMLE $\hat{\lambda}_n$ when h_n is divergent (yet at a slower rate than n). Assumptions 1–2, 3′, 4–7, and 10 of Lee (2004) are maintained, and the convergence rate of $\hat{\lambda}_n$ is $\sqrt{n/h_n}$.

The score function (evaluated at λ_0) based on the concentrated likelihood function, with the scaling parameter h_n/n imposed, is defined as

$$\psi_n = h_n r_{1n} - h_n g_{0n},$$

and the higher-order derivatives can be written as

$$\frac{\partial \psi_n(\lambda_0)}{\partial \lambda} = 2h_n r_{1n}^2 - h_n r_{2n} - h_n g_{1n}, \quad \frac{\partial^2 \psi_n(\lambda_0)}{\partial \lambda^2} = 8h_n r_{1n}^3 - 6h_n r_{1n} r_{2n} - 2h_n g_{2n},$$

where

$$\begin{aligned}
r_{1n} &= \frac{\boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{f}_n + \boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}{\boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{v}_n}, \\
r_{2n} &= \frac{\boldsymbol{f}'_n\boldsymbol{M}_n\boldsymbol{f}_n + \boldsymbol{v}'_n\boldsymbol{G}'_n\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n + 2\boldsymbol{v}'_n\boldsymbol{G}'_n\boldsymbol{M}_n\boldsymbol{f}_n}{\boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{v}_n}, \\
g_{in} &= \frac{1}{n}\mathrm{tr}(\boldsymbol{G}_n^{i+1}),\ i = 1, 2.
\end{aligned}$$

Using Appendix A ("Some Basic Properties") of Lee (2004), we can show that $g_{in} = O(h_n^{-1})$, $\boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n = O_P(n/h_n)$, $\boldsymbol{v}'_n\boldsymbol{G}'_n\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n = O_P(n/h_n)$, $\boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{f}_n = O_P(\sqrt{n/h_n})$, $\boldsymbol{v}'_n\boldsymbol{G}'_n\boldsymbol{M}_n\boldsymbol{f}_n = O_P(\sqrt{n/h_n})$, $\boldsymbol{v}'_n\boldsymbol{M}_n\boldsymbol{v}_n =$

$O_P(n)$, and $\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n - \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n) = O_P(\sqrt{n})$. Note that terms of order $O_P(n^{-1})$ are of order $o_P(h_n/n)$, since h_n is divergent. Thus, we can implement the following expansion

$$\begin{aligned}
h_n r_{1n} &= \frac{\frac{h_n}{n}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \frac{h_n}{n}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}{\frac{1}{n}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n} \\
&= \frac{\frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}{1 + \frac{1}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n - \frac{1}{n\sigma_0^2}\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \frac{k}{n}} \\
&= \frac{\frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}{1 + \frac{1}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n - \frac{1}{n\sigma_0^2}\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)} + O_P(n^{-1}) \\
&= \underbrace{\frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}_{O_P(1)} + \underbrace{\frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{f}_n}_{O_P\left((n/h_n)^{-1/2}\right)} + \underbrace{\frac{h_n}{n^2\sigma_0^4}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2})} + o_P\left(h_n/n\right).
\end{aligned}$$

Similarly, we have the following:

$$\begin{aligned}
h_n r_{2n} &= \underbrace{\frac{h_n}{n\sigma_0^2}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \frac{h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}_{O_P(1)} + \underbrace{\frac{2h_n}{n\sigma_0^2}\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{f}_n}_{O_P\left((n/h_n)^{-1/2}\right)} \\
&\quad + \underbrace{\frac{h_n}{n^2\sigma_0^4}\left(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\right)\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2})} + o_P\left(h_n/n\right),
\end{aligned}$$

$$\begin{aligned}
h_n r_{1n}^2 &= \underbrace{\frac{h_n}{n^2\sigma_0^4}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2}_{O_P(h_n^{-1})} + \underbrace{\frac{2h_n}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}_{O_P((nh_n)^{-1/2})} \\
&\quad + \underbrace{\frac{2h_n}{n^3\sigma_0^6}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2}h_n^{-1})} + o_P\left(h_n/n\right),
\end{aligned}$$

$$\begin{aligned}
h_n r_{1n}^3 &= \underbrace{\frac{h_n}{n^3\sigma_0^6}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3}_{O_P(h_n^{-2})} + \underbrace{\frac{3h_n}{n^3\sigma_0^6}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2}_{O_P(n^{-1/2}h_n^{-3/2})} \\
&\quad + \underbrace{\frac{3h_n}{n^4\sigma_0^8}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2}h_n^{-2})} + o_P\left(h_n/n\right),
\end{aligned}$$

$$\begin{aligned}
h_n r_{1n} r_{2n} &= \underbrace{\frac{h_n}{n^2\sigma_0^4}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O_P(h_n^{-1})} \\
&\quad + \underbrace{\frac{2h_n}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n + \frac{h_n}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n + \boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O_P((nh_n)^{-1/2})}
\end{aligned}$$

$$+\underbrace{\frac{2h_n}{n^3\sigma_0^6}(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2}h_n^{-1})}+o_P\left(h_n/n\right),$$

$$\begin{aligned}
h_n^2r_{1n}^2 &= \underbrace{\frac{h_n^2}{n^2\sigma_0^4}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2}_{O_P(1)}+\underbrace{\frac{2h_n^2}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n}_{O_P((n/h_n)^{-1/2})}+\underbrace{\frac{h_n^2}{n^2\sigma_0^4}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{f}_n\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n}_{O_P(h_n/n)} \\
&\quad+\underbrace{\frac{2h_n^2}{n^3\sigma_0^6}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2})}+o_P\left(h_n/n\right),
\end{aligned}$$

$$\begin{aligned}
h_n^2r_{1n}^3 &= \underbrace{\frac{h_n^2}{n^3\sigma_0^6}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3}_{O_P(h_n^{-1})}+\underbrace{\frac{3h_n^2}{n^3\sigma_0^6}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2}_{O_P((nh_n)^{-1/2})} \\
&\quad+\underbrace{\frac{3h_n^2}{n^4\sigma_0^8}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2}h_n^{-1})}+o_P\left(h_n/n\right),
\end{aligned}$$

$$\begin{aligned}
h_n^2r_{1n}r_{2n} &= \underbrace{\frac{h_n^2}{n^2\sigma_0^4}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\left(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\right)}_{O_P(1)} \\
&\quad+\underbrace{\frac{2h_n^2}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n+\frac{h_n^2}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n\left(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\right)}_{O_P((n/h_n)^{-1/2})} \\
&\quad+\underbrace{\frac{2h_n^2}{n^2\sigma_0^4}\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{f}_n\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n}_{O_P(h_n/n)} \\
&\quad+\underbrace{\frac{2h_n^2}{n^3\sigma_0^6}\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\left(\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n+\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\right)\left[\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right]}_{O_P(n^{-1/2})}+o_P\left(h_n/n\right).
\end{aligned}$$

Then by substitution,

$$\begin{aligned}
\mathrm{E}(\psi_n) &= \underbrace{\frac{h_n}{n\sigma_0^2}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)-h_ng_{0n}}_{O(1)}+\underbrace{\frac{h_n}{n^2\sigma_0^4}\left[\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n)\right]}_{O(n^{-1/2})}+o(h_n/n) \\
&\equiv \mathrm{E}(\psi_n)_{,O(1)}+\mathrm{E}(\psi_n)_{,O(n^{-1/2})}+o(h_n/n),
\end{aligned}$$

$$\mathrm{E}(H_n\psi_n) = \underbrace{h_n^2g_{0n}g_{1n}+\frac{h_n^2g_{0n}}{n\sigma_0^2}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n-\frac{h_n^2g_{1n}}{n\sigma_0^2}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O(1)}$$

$$
\begin{aligned}
& \underbrace{+\frac{h_n g_{0n}}{n\sigma_0^2}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n) - \frac{h_n^2}{n^2\sigma_0^4}\left[\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n) + \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\right]}_{O(1)} \\
& \underbrace{+\frac{2h_n^2}{n^3\sigma_0^6}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3] - \frac{2h_n^2 g_{0n}}{n^2\sigma_0^4}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2]}_{O\left(h_n^{-1}\right)} \\
& \underbrace{+\frac{-2h_n^2}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{G}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n) - \frac{h_n^2}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O((n/h_n)^{-1/2})} \\
& \underbrace{+\frac{-2h_n^2}{n^3\sigma_0^6}\left\{\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) + \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)\right\}}_{O(n^{-1/2})} \\
& \underbrace{+\frac{2h_n^2}{n^3\sigma_0^6}\left\{\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n) + \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n)\right\}}_{O(n^{-1/2})} \\
& \underbrace{+\frac{h_n^2 g_{1n}}{n^2\sigma_0^4}\left[\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n) - \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)\right]}_{O(n^{-1/2})} \\
& \underbrace{+\frac{h_n^2 g_{0n}}{n^2\sigma_0^4}\left[\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n)\right]}_{O(n^{-1/2})} \\
& \underbrace{+\frac{-2h_n^2}{n^2\sigma_0^4}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{f}_n\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n)}_{O(h_n/n)} + \underbrace{\frac{6h_n^2}{n^3\sigma_0^6}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}[\boldsymbol{v}_n(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2] - \frac{4h_n^2 g_{0n}}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O((nh_n)^{-1/2})} \\
& \underbrace{+\frac{6h_n^2}{n^4\sigma_0^8}\left\{\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3]\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right) - \mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n]\right\}}_{O(n^{-1/2}h_n^{-1})} \\
& \underbrace{+\frac{4h_n^2 g_{0n}}{n^3\sigma_0^6}\left\{\mathrm{E}[\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2] - \mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2]\right\}}_{O(n^{-1/2}h_n^{-1})} + o(h_n/n) \\
\equiv \; & \mathrm{E}(H_n\psi_n)_{,O(1)} + \mathrm{E}(H_n\psi_n)_{,O\left(h_n^{-1}\right)} + \mathrm{E}(H_n\psi_n)_{,O((n/h_n)^{-1/2})} + \mathrm{E}(H_n\psi_n)_{,O(n^{-1/2})} + \mathrm{E}(H_n\psi_n)_{,O(h_n/n)} \\
& + \mathrm{E}(H_n\psi_n)_{,O((nh_n)^{-1/2})} + \mathrm{E}(H_n\psi_n)_{,O(n^{-1/2}h_n^{-1})} + o(h_n/n),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\psi_n^2) \; = \; & \underbrace{h_n g_{0n}^2 + \frac{h_n^2}{n^2\sigma_0^4}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2] - \frac{2h_n^2 g_{0n}}{n\sigma_0^2}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O(1)} \\
& \underbrace{+\frac{h_n^2}{n^2\sigma_0^4}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{f}_n\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{v}_n)}_{O(h_n/n)} + \underbrace{\frac{2h_n^2}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O((n/h_n)^{-1/2})}
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\frac{2h_n^2}{n^3\sigma_0^6}\left\{\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2]\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n]\right\}}_{O(n^{-1/2})} \\
&+\underbrace{\frac{-2h_n^2 g_{0n}}{n^2\sigma_0^4}\left[\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n)\right]}_{O(n^{-1/2})}+o(h_n/n) \\
\equiv\ & \mathrm{E}(\psi_n^2)_{,O(1)}+\mathrm{E}(\psi_n^2)_{,O(h_n/n)}+\mathrm{E}(\psi_n^2)_{,O((n/h_n)^{-1/2})}+\mathrm{E}(\psi_n^2)_{,O(n^{-1/2})}+o(h_n/n),
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_n =\ & \underbrace{h_n g_{1n}+\frac{h_n}{n\sigma_0^2}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n+\frac{h_n}{n\sigma_0^2}\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O(1)}+\underbrace{\frac{-2h_n}{n^2\sigma_0^4}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2]}_{O(h_n^{-1})}+\underbrace{\frac{-4h_n}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O((nh_n)^{-1/2})} \\
&+\underbrace{\frac{h_n}{n^2\sigma_0^4}\left\{\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n)\right\}}_{O(n^{-1/2})} \\
&+\underbrace{\frac{4h_n}{n^3\sigma_0^6}\left\{\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n]-\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2]\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)\right\}}_{O(n^{-1/2}h_n^{-1})}+o(h_n/n) \\
\equiv\ & \Sigma_{n,O(1)}+\Sigma_{n,O(h_n^{-1})}+\Sigma_{n,O((nh_n)^{-1/2})}+\Sigma_{n,O(n^{-1/2})}+\Sigma_{n,O(n^{-1/2}h_n^{-1})}+o(h_n/n),
\end{aligned}
$$

$$
\begin{aligned}
D_n =\ & \underbrace{-2h_n g_{2n}}_{O(1)}+\underbrace{\frac{-6h_n}{n^2\sigma_0^4}\left[\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)+\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\right]}_{O(h_n^{-1})}+\underbrace{\frac{8h_n}{n^3\sigma_0^6}\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3]}_{O(h_n^{-2})} \\
&+\underbrace{\frac{-12h_n}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{G}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)-\frac{6h_n}{n^2\sigma_0^4}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}(\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)}_{O((nh_n)^{-1/2})} \\
&+\underbrace{\frac{12h_n}{n^3\sigma_0^6}\boldsymbol{f}_n'\boldsymbol{M}_n\boldsymbol{f}_n\left[\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)-\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)\right]}_{O(n^{-1/2}h_n^{-1})} \\
&+\underbrace{\frac{12h_n}{n^3\sigma_0^6}\left[\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)-\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{G}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)\right]}_{O(n^{-1/2}h_n^{-1})} \\
&+\underbrace{\frac{24h_n}{n^3\sigma_0^6}\boldsymbol{f}_n'\boldsymbol{M}_n\mathrm{E}[\boldsymbol{v}_n(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^2]}_{O(n^{-1/2}h_n^{-3/2})}+\underbrace{\frac{24h_n}{n^4\sigma_0^8}\left\{\mathrm{E}[(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3]\mathrm{E}\left(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n\right)-\mathrm{E}[\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{v}_n(\boldsymbol{v}_n'\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{v}_n)^3]\right\}}_{O(n^{-1/2}h_n^{-2})} \\
&+o\left(h_n/n\right) \\
\equiv\ & D_{n,O(1)}+D_{n,O(h_n^{-1})}+D_{n,O(h_n^{-2})}+D_{n,O((nh_n)^{-1/2})}+D_{n,O(n^{-1/2}h_n^{-1})}+D_{n,O(n^{-1/2}h_n^{-3/2})} \\
&+D_{n,O(n^{-1/2}h_n^{-2})}+o\left(h_n/n\right).
\end{aligned}
$$

Also, note that

$$
\Sigma_n^{-i} = \left[\Sigma_{n,O(1)}+\Sigma_{n,O(h_n^{-1})}+\Sigma_{n,O((nh_n)^{-1/2})}+\Sigma_{n,O(n^{-1/2})}+\Sigma_{n,O(n^{-1/2}h_n^{-1})}\right]^{-i}+o(h_n/n)
$$

$$\equiv \quad \Sigma_n^{(-i)} + o(h_n/n), \; i = 1, 2, 3.$$

We may do some further expansion of $\Sigma_n^{(-i)}$, but the above expansion shall suffice for our purpose, as all the terms in the expansion of Σ_n are known.

Thus, the approximate bias of $\hat{\lambda}_n$ should read

$$\begin{aligned}
\mathrm{B}(\hat{\lambda}_n) \quad = \quad & 2\Sigma_n^{(-1)} \left[\mathrm{E}(\psi_n)_{,O(1)} + \mathrm{E}(\psi_n)_{,O(n^{-1/2})} \right] + \Sigma_n^{(-2)} \left[\mathrm{E}(H_n\psi_n)_{,O(1)} + \mathrm{E}(H_n\psi_n)_{,O\left(h_n^{-1}\right)} \right. \\
& + \mathrm{E}(H_n\psi_n)_{,O((n/h_n)^{-1/2})} + \mathrm{E}(H_n\psi_n)_{,O(n^{-1/2})} + \mathrm{E}(H_n\psi_n)_{,O(h_n/n)} + \mathrm{E}(H_n\psi_n)_{,O((nh_n)^{-1/2})} \\
& \left. + \mathrm{E}(H_n\psi_n)_{,O(n^{-1/2}h_n^{-1})} \right] + \frac{1}{2}\Sigma_n^{(-3)} \left[D_{n,O(1)} + D_{n,O(h_n^{-1})} + D_{n,O(h_n^{-2})} \right] \left[\mathrm{E}(\psi_n^2)_{,O(1)} \right. \\
& \left. + \mathrm{E}(\psi_n^2)_{,O(h_n/n)} + \mathrm{E}(\psi_n^2)_{,O((n/h_n)^{-1/2})} + \mathrm{E}(\psi_n^2)_{,O(n^{-1/2})} \right] + \frac{1}{2}\Sigma_n^{(-3)} \left[D_{n,O((nh_n)^{-1/2})} \right. \\
& \left. + D_{n,O(n^{-1/2}h_n^{-1})} + D_{n,O(n^{-1/2}h_n^{-3/2})} + D_{n,O(n^{-1/2}h_n^{-2})} \right] \mathrm{E}(\psi_n^2)_{,O(1)}.
\end{aligned}$$

Note all the expectations are in terms of quadratic and linear forms in the nonnormal vector $\boldsymbol{v}_n$, up to order 4. The results of Bao and Ullah (2010, “Expectation of quadratic forms in normal and nonnormal variables with applications”, *Journal of Statistical Planning and Inference*, 140) can be used directly. One may be tempted to do further substitutions and simplify the express for $\mathrm{B}(\hat{\lambda}_n)$. It may also turn out after simplification, some terms may be of order $o(h_n/n)$ and thus should drop out from $\mathrm{B}(\hat{\lambda}_n)$. (We suspect that terms involving cumulants $\gamma_3, \cdots, \gamma_6$ are of order $o(h_n/n)$.) The cost is more tedious algebra and we do not pursue further here.