Estimating a Spatial Autoregressive Model with Autoregressive Disturbances Based on the Indirect Inference Principle

Yong Bao*
ybao@purdue.edu
Department of Economics
Purdue University

Xiaotian Liu
liu2004@purdue.edu
Department of Economics
Purdue University

Abstract

This paper proposes a new estimation procedure for the first-order spatial autoregressive model, where the disturbance term also follows a first-order autoregression and its innovations may be heteroscedastic. The estimation procedure is based on the principle of indirect inference that matches the ordinary least squares estimator of the two spatial autoregressive coefficients (one in the outcome equation and the other in the disturbance equation) with its approximate analytical expectation. The resulting estimator is shown to be consistent, asymptotically normal, and robust to unknown heteroscedasticity. Monte Carlo experiments are provided to show its finite-sample performance in comparison with existing estimators that are based on the generalized method of moments. The new estimation procedure is applied to empirical studies on teenage pregnancy rates and Airbnb accommodation prices.

Key Words: spatial autoregressive model; indirect inference; ordinary least squares

JEL classification: C21, C31

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*Corresponding author: Department of Economics, Purdue University, 403 W State St, West Lafayette, IN 47907, USA.

INTRODUCTION

Spatial autoregressive (SAR) models have been widely used in many disciplines of social sciences by extending the notion of autocorrelation from the traditional time domain to space. Spatial correlation may arise from different sources such as strategic interaction, spill-over, copycatting, and general equilibrium effects, to name just a few. In this framework, space can be defined not only in the geographical sense but also from economic and social perspectives. A classical treatment of this subject is Cliff and Ord (1981) and a more recent one is LeSage and Pace (2009).

This paper considers the first-order SAR model with first-order autoregressive disturbances (SARAR(1,1) for short), which extends the popular first-order SAR (SAR(1)) model by allowing for a more general structure of spatial correlation that may originate from both the observable and unobservable. Under the assumption of homoscedastic error innovations, Kelejian and Prucha (1998) proposed a generalized spatial two-stage least squares (GS2SLS) procedure to estimate SARAR(1,1). Lee (2003) proposed the best GS2SLS by replacing the IV (instrumental variables) matrix of the GS2SLS estimator in Kelejian and Prucha (1998) with the asymptotically optimal one. Lee and Liu (2010) discussed the generalized method of moments (GMM) and proposed the best GMM estimator. Burridge (2012) discussed how to solve for the quasi maximum likelihood (QML) estimator for the SARAR(1,1) model by a numerical search algorithm and recently Liu and Lee (2019) derived the asymptotic properties of the QML estimator in SARAR(1,1). Kelejian and Prucha (2010) extended their GS2SLS to allow for heteroscedasticity in error innovations and Jin and Lee (2019) compared the generalized empirical likelihood (GEL) and GMM estimators in this general framework. Taşpınar et al. (2019) considered various ways, robust to heteroscedasticity, to improve the finite-sample properties of the GMM estimator in SARAR(1,1).[1] In comparison with the QML, the IV/GMM approach enjoys not only computational simplicity (in that it does not need to calculate the determinants of matrices involving the spatial weight matrices, which is required for the QML) but also robustness against departure from homoscedasticity.

The existing IV/GMM literature appears to rely on the so-called instrumental variables, possibly together with some linear and quadratic moment conditions (associated with the error term), to estimate SARAR models. Different choices of IV and moment conditions can result in different estimation methods with different numerical optimization procedures. They are also directly related to the complexity of the resulting asymptotic variance of the corresponding estimator. This paper takes a different approach that does not rely on IV or moment conditions. In particular, it estimates model parameters by matching the simple ordinary least squares (OLS) estimator of the two spatial autoregressive coefficients (one in the outcome equation and the other in the disturbance equation) with its approximate analytical expectation. This approach is largely in line with the indirect inference (II) procedure of Gouriéroux et al. (1993) and Smith (1993). However, the original II is simulation-based in the sense that the relevant expectation is approximated by the average of simulated estimates and one needs to make distributional assumptions on the pseudo error term in simulations. Kyriacou et al. (2017) studied the SAR(1) model by working out the approximate expectation of the OLS estimator of the SAR coefficient and then matching with the inconsistent OLS estimator to "solve" for the SAR parameter. Nevertheless, their model does not include exogenous regressors and the disturbance term is serially uncorrelated and homoscedastic. Recently, Kyriacou et al. (2019) and Bao et al. (2020) have extended the SAR(1) to include exogenous regressors with possibly heteroscedastic errors.[2] This paper considers a more general framework where disturbances are spatially correlated and innovations of the error process are heteroscedastic. Just as an ARMA process relative to an AR process in time series, a SARAR model, compared with a SAR specification, is able to describe a richer spectrum of interactions and heterogeneity among cross-sectional units. However, the presence of spatial correlation in the error term introduces nontrivial technical difficulty. First of all, one cannot simply ignore the correlation in the error process to estimate the outcome equation by following the approach of Kyriacou et al. (2019) or Bao et al. (2020) that is robust to error heteroscedasticity. The binding function (pertaining to the SAR parameter in the outcome equation) involves the SAR parameter in the error process, so one cannot solve the binding function to estimate the SAR parameter in the outcome equation. Secondly, the traditional Cochrane-Orcutt procedure that aims for dealing with error correlation does not work, since the OLS estimator of the SAR parameter in the error process is not consistent even if one knows the SAR parameter in the outcome equation. The novelty of this paper is to design two binding functions, one for each of the SAR parameters such that both are expressed in terms of the observable data. The first binding function related to the SAR parameter in the outcome equation depends on the SAR parameter in the error process. The second binding function, since it is built from a consistent residual vector, which in turn depends on the SAR parameter in the error process, involves both SAR parameters. Given the observable sample data, the two resulting binding functions constitute a system of two equations in terms of the two unknown SAR parameters.

Similar to the IV/GMM estimator, the II estimator proposed in this paper is computationally simpler relative to the QML estimator and is robust to heteroscedasticity. In comparison with the IV/GMM estimator, the II estimator possesses three salient features. Firstly, it is free of the choice of IV or moment conditions. This may be relevant when one is unsure about the choices of IV and moment conditions or when one is daunted by the complexity of the optimal weight matrix, as it involves the error innovation variance matrix and this may produce some undesirable consequences in the numerical optimization when the estimated variance matrix is used in the weight matrix. Secondly, the II procedure may enjoy some degree of computational advantage. It is based on a 2-dimensional numerical search since it solves for the two spatial autoregressive parameters (λ and ρ, appearing in the outcome equation and the error process, respectively) using two sample binding functions established from the simple OLS procedure. Once the two spatial parameters are estimated consistently, the coefficient vector $\boldsymbol{\beta}$ associated with exogenous regressors in the outcome equation can be easily estimated by the usual OLS procedure. The GS2SLS of Kelejian and Prucha (1998, 2010) involves two steps that estimate the spatial autoregressive parameters separately. In the first step, λ and $\boldsymbol{\beta}$ are estimated by 2SLS based on some IV. In the second step, ρ is estimated by GMM using some quadratic moment conditions. These moment conditions are designed by some careful choices of the relevant matrices appearing in the quadratic forms in the error innovations. (And such careful choices also deal with heteroscedasticity.) The GMM estimator in Lee and Liu (2010) and Jin and Lee (2019) estimates λ, ρ, and $\boldsymbol{\beta}$ jointly by using some linear and quadratic moment conditions associated with the error innovations. The numerical search in GMM is over a $(k+2)$-dimensional parameter space, where k is the dimension of $\boldsymbol{\beta}$. The optimal weight matrix (in formulating the quadratic form of the objective function in GMM and in the second step of GS2SLS) involves the error innovation variance matrix and to make it feasible one typically needs to estimate it based on some initial consistently estimated parameters. Thirdly, the II procedure estimates jointly λ and ρ first and then $\boldsymbol{\beta}$ is estimated by the usual OLS plug-in procedure. So essentially, it is also a two-step procedure. Recall that the GS2SLS of Kelejian and Prucha (1998, 2010) estimates λ and $\boldsymbol{\beta}$ first by 2SLS and then ρ by GMM. Lee (2007) and Yang (2015) emphasized that the spatial coefficients are the main source of bias in model estimation and the main cause of difficulty in bias correction in SAR models. In fact, Monte Carlo experiments in this paper show that in the first step of GS2SLS, it can happen that both λ and (some elements of) $\boldsymbol{\beta}$ may be estimated with relatively large magnitudes of biases. This happens because the first step of GS2SLS totally ignores the degree of spatial correlation in the error term. The II procedure on the other hand takes care of the two spatial coefficients jointly in one step.

The plan of this paper is as follows. Section 2 describes the model specification and the main assumptions used in this paper. Section 3 discusses the estimation procedure. In particular, the asymptotic behavior of the (inconsistent) OLS estimator is discussed and then the II estimation procedure is described and its asymptotic properties are provided. Section 4 reports results from Monte Carlo experiments. It shows that the II estimator performs better than the GS2SLS estimator of Kelejian and Prucha (1998, 2010) and the GMM estimator of Jin and Lee (2019) in finite samples when a sparse county contiguity matrix is used. It is found that the GS2SLS- and GMM-based inference procedures can give rise to severe size distortions when the degree of spatial correlation in the error process is high. In contrast, the II-based t-test delivers excellent finite-sample size performance. When the spatial weight matrices are relatively dense, however, the three estimators can perform poorly in small samples. Section 5 contains two empirical studies, one on teenage pregnancy rates and the other on Airbnb listing prices. Section 6 concludes. Technical details and additional simulation results are collected in the appendix.

Throughout, tr denotes matrix trace operator, $\mathrm{Dg}(\boldsymbol{a}_n)$ denotes a diagonal matrix with the vector $\boldsymbol{a}_n$ spanning the main diagonal, and $\mathrm{Dg}(\boldsymbol{A}_n)$ is a diagonal matrix that collects the diagonals elements of the square matrix $\boldsymbol{A}_n$. The subscript 0 is used to signify the true parameter value.

MODEL SPECIFICATION

Consider the following SARAR(1,1) model

$$\boldsymbol{y}_n = \boldsymbol{X}_n\boldsymbol{\beta} + \lambda\boldsymbol{W}_n\boldsymbol{y}_n + \boldsymbol{u}_n, \qquad \boldsymbol{u}_n = \rho\boldsymbol{M}_n\boldsymbol{u}_n + \boldsymbol{v}_n, \tag{1}$$

where $\boldsymbol{y}_n$ is an $n \times 1$ vector of observations on the dependent variable, $\boldsymbol{X}_n$ is an $n \times k$ matrix of observations on k exogenous deterministic regressors with coefficient vector $\boldsymbol{\beta}$, $\boldsymbol{u}_n$ is an $n \times 1$ vector of regression disturbances, $\boldsymbol{v}_n$ is an $n \times 1$ vector of innovations, λ and ρ are the spatial autoregressive coefficients, and $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ are $n \times n$ matrices of spatial weights.

For the ease of presentation, let $\boldsymbol{S}_n(\lambda) = \boldsymbol{I}_n - \lambda \boldsymbol{W}_n$, $\boldsymbol{R}_n(\rho) = \boldsymbol{I}_n - \rho \boldsymbol{M}_n$, $\boldsymbol{G}_n(\lambda) = \boldsymbol{W}_n \boldsymbol{S}_n^{-1}(\lambda)$, $\boldsymbol{F}_n(\rho) = \boldsymbol{M}_n \boldsymbol{R}_n^{-1}(\rho)$, and $\boldsymbol{H}_n(\rho) = \boldsymbol{I}_n - \boldsymbol{R}_n(\rho)\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'(\rho)$. When a matrix is presented without its argument, it means that it is evaluated at the true parameter value. That is, $\boldsymbol{S}_n = \boldsymbol{S}_n(\lambda_0)$, $\boldsymbol{R}_n = \boldsymbol{R}_n(\rho_0)$, $\boldsymbol{G}_n = \boldsymbol{G}_n(\lambda_0)$, $\boldsymbol{F}_n = \boldsymbol{F}_n(\rho_0)$, and $\boldsymbol{H}_n = \boldsymbol{H}_n(\rho_0)$. With such a set of notation, the equilibrium solution of the process is $\boldsymbol{y}_n = \boldsymbol{S}_n^{-1}\boldsymbol{X}_n\boldsymbol{\beta}_0 + \boldsymbol{S}_n^{-1}\boldsymbol{R}_n^{-1}\boldsymbol{v}_n$. Throughout, the following assumptions are made.

Assumption 1. *(i) The row and column sums of $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ are bounded uniformly in absolute value. (ii) The diagonal elements of $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ are all zero.*

Assumption 2. *(i) $\boldsymbol{S}_n^{-1}$ and $\boldsymbol{R}_n^{-1}$ exist. (ii) The row and column sums of $\boldsymbol{S}_n^{-1}$ and $\boldsymbol{R}_n^{-1}$ are bounded uniformly in absolute value.*

Assumption 3. *For $1 \leq i \leq n$, the innovation terms $v_{i,n}$ in $\boldsymbol{v}_n = (v_{1,n}, \cdots, v_{n,n})'$ are mutually independent with $\mathrm{E}(v_{i,n}) = 0$, $\mathrm{E}(v_{i,n}^2) = \sigma_{i,n}^2$, and $\mathrm{E}(|v_{i,n}|^{4+\delta}) < \infty$ for some positive constant δ.*

Assumption 4. *(i) λ_0 and ρ_0 are contained in compact parameter spaces Λ and P, respectively. (ii) For any admissible $\lambda \in \Lambda$ and $\rho \in \mathrm{P}$, the row and column sums of $\boldsymbol{S}_n^{-1}(\lambda)$ and $\boldsymbol{R}_n^{-1}(\rho)$ are bounded uniformly in absolute value.*

Assumption 5. *(i) The elements of $\boldsymbol{X}_n$ are uniformly bounded. (ii) The limit*

$$\lim_{n\to\infty} n^{-1}(\boldsymbol{X}_n, \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0)'(\boldsymbol{X}_n, \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0)$$

exists and is nonsingular.

Assumption 6. *Let $\boldsymbol{\Sigma}_n = \mathrm{Dg}(\sigma_{1,n}^2, ..., \sigma_{n,n}^2)$. Then*

$$\boldsymbol{\Xi} = \begin{pmatrix} \xi_1 & \xi_{12} \\ \xi_{12} & \xi_2 \end{pmatrix}$$

exists and is positive definite, where

$$\begin{aligned}
\xi_1 &= \lim_{n\to\infty} \frac{n\{\mathrm{tr}[\boldsymbol{\Sigma}_n\boldsymbol{E}_n\boldsymbol{\Sigma}_n(\boldsymbol{E}_n + \boldsymbol{E}_n')] + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\}}{[\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0]^2}, \\
\xi_2 &= \lim_{n\to\infty} \frac{n\mathrm{tr}[\boldsymbol{\Sigma}_n\boldsymbol{L}_n\boldsymbol{\Sigma}_n(\boldsymbol{L}_n + \boldsymbol{L}_n')]}{[\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F'}_n\boldsymbol{F}_n)]^2}, \\
\xi_{12} &= \lim_{n\to\infty} \frac{n\mathrm{tr}[\boldsymbol{\Sigma}_n\boldsymbol{E}_n\boldsymbol{\Sigma}_n(\boldsymbol{L}_n + \boldsymbol{L}_n')]}{\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F'}_n\boldsymbol{F}_n)[\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0]},
\end{aligned}$$

in which $\boldsymbol{E}_n = \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} - \mathrm{Dg}(\boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1})$ *and* $\boldsymbol{L}_n = \boldsymbol{F}_n - \mathrm{Dg}(\boldsymbol{F}_n)$.

Assumption 1.(ii) is a normalization rule often assumed in the literature to exclude "self influence." Assumptions 1.(i), 2 and 4 limit the degree of spatial dependency and are originated by Kelejian and Prucha (1998). Assumption 3 is the same as in Kelejian and Prucha (2010) and Jin and Lee (2019), which allows for heteroscedasticity in the innovations. If one further assumes that the innovations are i.i.d., then the QML can be used. In Kyriacou et al. (2017), there are no exogenous regressors and the disturbances contain no SAR structure and are i.i.d. Their Monte Carlo experiments showed that their II estimator is comparable to the QML estimator while losing efficiency in some cases. One would expect that the II estimator introduced in this paper may lose efficiency relative to the QML estimator if the innovations are i.i.d. Lee (2002) emphasized that Assumption 5.(ii) is related to an identification condition for estimation in the least squares and IV frameworks and it rules out possible multicollinearities among $\boldsymbol{X}_n$ and $\boldsymbol{G}_n \boldsymbol{X}_n \boldsymbol{\beta}_0$ for large n. Assumption 6 is related to the asymptotic variance of the II estimator.

ESTIMATION PROCEDURE

This section provides the main results. The OLS estimator is briefly discussed first and its asymptotic distribution, when properly recentered, is presented. Since the recentering terms involve the unknown model parameters as well as the variance matrix of the error vector, the recentered OLS estimator is not usable in practice. The II estimator solves for the unknown parameters by utilizing two binding functions that do not rely on the unknown variance matrix. It is then shown that the II estimator is consistent and asymptotically normal.

The OLS Estimator

If the true value of ρ is known, the Cochrane-Orcutt-type transformation to (1) yields

$$\boldsymbol{R}_n \boldsymbol{y}_n = \boldsymbol{R}_n \boldsymbol{X}_n \boldsymbol{\beta} + \lambda \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n + \boldsymbol{v}_n. \tag{2}$$

The OLS estimator of λ_0 for the transformed model (2), depending explicitly on the true value ρ_0, is given by

$$\hat{\lambda}(\rho_0) = \frac{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{y}_n}{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n} = \lambda_0 + \frac{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{v}_n}{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n}. \tag{3}$$

The probability limit of the ratio $\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{v}_n/\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n$ is non-zero so the OLS estimator of λ_0, even if ρ_0 is given, is not consistent. One cannot follow Kyriacou et al. (2017) to seek a consistent estimator of λ_0 by building a binding function that takes the (approximate) expectation of the ratio as (3) depends on the unknown value ρ_0, so is the resulting binding function. One cannot solve for λ without knowing ρ.

The strategy in this paper is to build another binding function based on the OLS estimator of ρ_0 that is constructed from a consistent residual vector, namely,

$$\hat{\rho}(\lambda_0,\rho_0) = \frac{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n}{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n} = \rho_0 + \frac{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n\tilde{\boldsymbol{v}}_n}{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n}, \tag{4}$$

where $\tilde{\boldsymbol{u}}_n = \tilde{\boldsymbol{u}}_n(\lambda_0,\rho_0) = \boldsymbol{R}_n^{-1}\tilde{\boldsymbol{v}}_n$, $\tilde{\boldsymbol{v}}_n = \tilde{\boldsymbol{v}}_n(\lambda_0,\rho_0) = \boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n = \boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{R}_n\boldsymbol{X}_n\tilde{\boldsymbol{\beta}}_n$, and $\tilde{\boldsymbol{\beta}}_n = \tilde{\boldsymbol{\beta}}_n(\lambda_0,\rho_0) = (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$.

The OLS estimator, as defined in (3) and (4), is not feasible, since it involves the unknown λ_0 and ρ_0. It is not consistent either. However, one can properly recenter $\hat{\lambda}(\rho_0)$ and $\hat{\rho}(\lambda_0,\rho_0)$ and the resulting recentered estimator, though still infeasible, achieves consistency. One choice of the re-centering term for $\hat{\lambda}(\rho_0)$ is $c_\lambda = \mathrm{E}(\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{v}_n)/\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n = \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1})/\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n$.[3] One can show that $\sqrt{n}(\hat{\lambda}(\rho_0)-\lambda_0-c_\lambda)$ is asymptotically equivalent to $\sqrt{n}[\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{y}_n - \mathrm{E}(\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{y}_n)]/\mathrm{E}(\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n)$. Substituting $\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n = \boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0 + \boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}\boldsymbol{v}_n$, one can see that the random parts of $\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{y}_n$ and $\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{y}_n$ are linear and quadratic forms in the random vector $\boldsymbol{v}_n$. Then from Lemma **??** (in the appendix), $\sqrt{n}(\hat{\lambda}(\rho_0)-\lambda_0-c_\lambda)$ converges to a zero-mean normal random variable. For $\hat{\rho}(\lambda_0,\rho_0)$, the re-centering term is not obvious. By using $\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n\tilde{\boldsymbol{v}}_n - \boldsymbol{u}_n'\boldsymbol{M}_n\boldsymbol{v}_n = O_p(1)$ and $\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n - \boldsymbol{u}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{u}_n = O_p(1)$ (see Lemma **??**), where $\boldsymbol{u}_n'\boldsymbol{M}_n\boldsymbol{v}_n = \boldsymbol{v}_n'\boldsymbol{F}_n\boldsymbol{v}_n$ with $\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{F}_n\boldsymbol{v}_n) = \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n)$ and $\boldsymbol{u}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{u}_n = \boldsymbol{v}_n'\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{v}_n = \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$, the re-centering term for $\hat{\rho}_n(\lambda_0,\rho_0)$ can can be chosen as $c_\rho = \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n)/\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$. Some algebra shows that $\sqrt{n}(\hat{\rho}_n-\rho_0-c_\rho)$ is asymptotically equivalent to $\sqrt{n}[\boldsymbol{v}_n'\boldsymbol{F}_n\boldsymbol{v}_n - \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n)]/\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)$, which converges to a zero-mean normal random variable.

The correction terms ($c_\lambda = \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1})/\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n$ for $\hat{\lambda}(\rho_0)$ and $c_\rho = \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n)/\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$ for $\hat{\rho}_n(\lambda_0,\rho_0)$) involve, as usual, the unknown parameters. Moreover, they contain the annoying $\boldsymbol{\Sigma}_n$. Since $\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) = \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{D}_n) = \mathrm{E}(\boldsymbol{v}_n'\boldsymbol{D}_n\boldsymbol{v}_n)$, where $\boldsymbol{D}_n = \mathrm{Dg}(\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1})$, one may wonder whether replacing $\mathrm{E}(\boldsymbol{v}_n'\boldsymbol{D}_n\boldsymbol{v}_n)$

with $\tilde{\boldsymbol{v}}_n'\boldsymbol{D}_n\tilde{\boldsymbol{v}}_n = \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$ in the correction term for $\hat{\lambda}(\rho_0)$ can yield a useful asymptotic distribution result. (And similarly, replace $\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n)$ in the correction term for $\hat{\rho}_n(\lambda_0, \rho_0)$ with $\tilde{\boldsymbol{v}}_n'\boldsymbol{K}_n\tilde{\boldsymbol{v}}_n = \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$, where $\boldsymbol{K}_n = \mathrm{Dg}(\boldsymbol{F}_n)$.) It turns out the answer is positive.

Theorem 1. *Under Assumptions 1–6, the OLS estimator $(\hat{\lambda}(\rho_0), \hat{\rho}(\lambda_0, \rho_0))'$, as defined in (3) and (4), has the following asymptotic distribution:*

$$\sqrt{n}\begin{pmatrix} \hat{\lambda}(\rho_0) - \lambda_0 - \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n} \\ \hat{\rho}(\lambda_0, \rho_0) - \rho_0 - \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n} \end{pmatrix} \xrightarrow{d} \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Xi}). \tag{5}$$

Now the recentering terms involve only the sample data and model parameters, but not the nuisance matrix $\boldsymbol{\Sigma}_n$. This makes it feasible to design the II estimator that corrects the inconsistency of the original OLS estimator.

The II Estimator

The asymptotic distribution result (5) can be used to design an estimator of $(\lambda_0, \rho_0)'$ in the spirit of indirect inference by matching $(\hat{\lambda}(\rho_0), \hat{\rho}(\lambda_0, \rho_0))'$ with its (approximate) expectation. Recall $\hat{\lambda}(\rho_0) = \boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'(\rho_0)\boldsymbol{H}_n(\rho_0)\boldsymbol{R}_n(\rho_0)\boldsymbol{y}_n / \boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'(\rho_0)\boldsymbol{H}_n(\rho_0)\boldsymbol{R}_n(\rho_0)\boldsymbol{W}_n\boldsymbol{y}_n$ and from (5), $\hat{\lambda}(\rho_0)$ centers around

$$\lambda_0 + \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'(\lambda_0)\boldsymbol{R}_n'(\rho_0)\boldsymbol{H}_n(\rho_0)\boldsymbol{D}_n(\lambda_0, \rho_0)\boldsymbol{H}_n(\rho_0)\boldsymbol{R}_n(\rho_0)\boldsymbol{S}_n(\lambda_0)\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'(\rho_0)\boldsymbol{H}_n(\rho_0)\boldsymbol{R}_n(\rho_0)\boldsymbol{W}_n\boldsymbol{y}_n},$$

where the dependency of various matrices on $(\lambda_0, \rho_0)'$ is explicitly expressed. So a binding function for finding the true parameter value λ_0 is

$$\begin{aligned} b_{1n}(\lambda, \rho) = & \frac{\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{W}_n\boldsymbol{y}_n} \\ & - \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'(\lambda)\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{D}_n(\lambda, \rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{S}_n(\lambda)\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{W}_n\boldsymbol{y}_n} - \lambda. \end{aligned}$$

Of course, $b_{1n}(\lambda, \rho) = 0$ alone cannot solve for λ since it involves two unknowns. It has to be combined with a second binding function pertaining to ρ, which follows similarly:

$$b_{2n}(\lambda, \rho) = \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'(\lambda)\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n^{-1\prime}(\rho)\boldsymbol{F}_n(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{S}_n(\lambda)\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{S}_n'(\lambda)\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{F}_n'(\rho)\boldsymbol{F}_n(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{S}_n(\lambda)\boldsymbol{y}_n}$$

$$-\frac{\boldsymbol{y}_n'\boldsymbol{S}_n'(\lambda)\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{K}_n(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{S}_n(\lambda)\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{S}_n'(\lambda)\boldsymbol{R}_n'(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{F}_n'(\rho)\boldsymbol{F}_n(\rho)\boldsymbol{H}_n(\rho)\boldsymbol{R}_n(\rho)\boldsymbol{S}_n(\lambda)\boldsymbol{y}_n}-\rho.$$

The II estimator $(\hat{\lambda}_{II},\hat{\rho}_{II})'$ of $(\lambda,\rho)'$ is thus defined as the root of $\boldsymbol{b}_n(\lambda,\rho)=(b_{1n}(\lambda,\rho),b_{2n}(\lambda,\rho))'$.

Assumption 7. *For* $(\lambda,\rho)'\in\Lambda\times\mathrm{P}$, *(i)* $\Pr(\lim_{n\to\infty}\boldsymbol{b}_n(\lambda_0,\rho_0)=\mathbf{0})=1$ *and* $\Pr(\lim_{n\to\infty}\boldsymbol{b}_n(\lambda,\rho)\neq\mathbf{0})=1$ *for any* $(\lambda,\rho)'\neq(\lambda_0,\rho_0)'$, *(ii) the Jacobian* $\boldsymbol{B}_n(\lambda,\rho)$ *of* $\boldsymbol{b}_n(\lambda,\rho)$ *is nonsingular almost surely, and (iii)* $\boldsymbol{B}_n(\lambda_0,\rho_0)\overset{a.s.}{\to}\boldsymbol{B}$, *where* $\boldsymbol{B}$ *is nonsingular.*

Essentially, Assumption 7.(i) ensures the existence and uniqueness of the root of $\boldsymbol{b}_n(\lambda,\rho)$, at least in large samples.[4] Assumptions 7.(ii) and 7.(iii) are needed to derive the asymptotic distribution of the resulting II estimator.

Theorem 2. *For model (1), under Assumptions 1–7, the II estimator* $(\hat{\lambda}_{II},\hat{\rho}_{II})'$ *of* $(\lambda,\rho)'$, *defined as the root of* $\boldsymbol{b}_n(\lambda,\rho)$, *has the following asymptotic distribution,*

$$\sqrt{n}\begin{pmatrix}\hat{\lambda}_{II}-\lambda_0\\ \hat{\rho}_{II}-\rho_0\end{pmatrix}\overset{d}{\to}\mathrm{N}(\mathbf{0},\boldsymbol{\Gamma}),\tag{6}$$

where $\boldsymbol{\Gamma}=\boldsymbol{B}^{-1}\boldsymbol{\Xi}\boldsymbol{B}^{-1\prime}=((\gamma_\lambda,\gamma_{\lambda\rho})',(\gamma_{\lambda\rho},\gamma_\rho)')'$.

Once $(\lambda_0,\rho_0)'$ is estimated by $(\hat{\lambda}_{II},\hat{\rho}_{II})'$, one can estimate $\boldsymbol{\beta}_0$ by

$$\hat{\boldsymbol{\beta}}_{II}=(\boldsymbol{X}_n'\boldsymbol{R}_n'(\hat{\rho}_{II})\boldsymbol{R}_n(\hat{\rho}_{II})\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'(\hat{\rho}_{II})\boldsymbol{R}_n(\hat{\rho}_{II})\boldsymbol{S}_n(\hat{\lambda}_{II})\boldsymbol{y}_n.\tag{7}$$

Given that $\hat{\lambda}_{II}$ and $\hat{\rho}_{II}$ are consistent, $\hat{\boldsymbol{\beta}}_{II}$ is necessarily consistent. Its asymptotic variance, however, is different from the traditional OLS variance formula given the additional uncertainty introduced by $\hat{\lambda}_{II}$ and $\hat{\rho}_{II}$. The following theorem gives the joint asymptotic distribution of $\hat{\lambda}_{II}$, $\hat{\rho}_{II}$, and $\hat{\boldsymbol{\beta}}_{II}$.

Theorem 3. *For model (1), under Assumptions 1–7,*

$$\sqrt{n}\begin{pmatrix}\hat{\lambda}_{II}-\lambda_0\\ \hat{\rho}_{II}-\rho_0\\ \hat{\boldsymbol{\beta}}_{II}-\boldsymbol{\beta}_0\end{pmatrix}\overset{d}{\to}\mathrm{N}(\mathbf{0},\boldsymbol{V}),\tag{8}$$

where

$$\boldsymbol{V}=\begin{pmatrix}\boldsymbol{\Gamma}&\gamma'\\ \gamma&\boldsymbol{\Gamma}_{\boldsymbol{\beta}}\end{pmatrix},$$

is assumed to exist and be positive definite, and

$$
\begin{aligned}
\boldsymbol{\gamma} &= \left(\boldsymbol{\gamma}_{\boldsymbol{\beta}\lambda}, \boldsymbol{\gamma}_{\boldsymbol{\beta}\rho}\right), \\
\boldsymbol{\gamma}_{\boldsymbol{\beta}\lambda} &= \lim_{n\to\infty}\left\{\frac{nb_{11}^{(-1)}(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0}{\operatorname{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0}\right. \\
&\quad \left. - (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\gamma_\lambda\right\}, \\
\boldsymbol{\gamma}_{\boldsymbol{\beta}\rho} &= \lim_{n\to\infty}\left\{\frac{nb_{21}^{(-1)}(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0}{\operatorname{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0}\right. \\
&\quad \left. - (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\gamma_{\lambda\rho}\right\}, \\
\boldsymbol{\Gamma}_{\boldsymbol{\beta}} &= \lim_{n\to\infty}\Big[n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{\Sigma}_n\boldsymbol{R}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1} \\
&\quad + (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\gamma_\lambda \\
&\quad - \frac{nb_{11}^{(-1)}(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}}{\operatorname{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0} \\
&\quad - \left.\frac{nb_{11}^{(-1)}(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{\Sigma}_n\boldsymbol{R}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}}{\operatorname{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0}\right],
\end{aligned}
$$

in which $b_{ij}^{(-1)}$ *denotes the* (i, j)*-th element of* $\boldsymbol{B}^{-1}$.

In practice, one can estimate the asymptotic variance matrix $\boldsymbol{V}$ by replacing all the unknowns appearing in $\boldsymbol{\Gamma}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\Gamma}_{\boldsymbol{\beta}}$ with their consistent estimates and the limits with the sample analogues. Further, one may replace $\boldsymbol{\Sigma}_n$ with $\hat{\boldsymbol{\Sigma}}_n = \operatorname{Dg}(\hat{\boldsymbol{v}}_n\hat{\boldsymbol{v}}_n')$, where $\hat{\boldsymbol{v}}_n = \hat{\boldsymbol{H}}_n\hat{\boldsymbol{R}}_n\hat{\boldsymbol{S}}_n\boldsymbol{y}_n$ with $\hat{\boldsymbol{H}}_n = \boldsymbol{H}_n(\hat{\rho}_{II})$, $\hat{\boldsymbol{R}}_n = \boldsymbol{R}_n(\hat{\rho}_{II})$, and $\hat{\boldsymbol{S}}_n = \boldsymbol{S}_n(\hat{\lambda}_{II})$.[5] So the estimated $\boldsymbol{V}$ may be denoted by $\hat{\boldsymbol{V}}_n = \hat{\boldsymbol{V}}_n(\hat{\lambda}_{II}, \hat{\rho}_{II}, \hat{\boldsymbol{\beta}}_{II}, \boldsymbol{y}_n, \boldsymbol{X}_n)$.

SIMULATION RESULTS

In this section, Monte Carlo simulations are conducted to illustrate the finite-sample performance of the proposed II estimator, in comparison with the GMM estimator of Jin and Lee (2019) and the GS2SLS estimator of Kelejian and Prucha (2010).[6] The spatial weight matrix $\boldsymbol{W}_n$ is the row-normalized county contiguity matrix used in Lin and Lee (2010) with $n = 761$ and $\boldsymbol{M}_n = \boldsymbol{W}_n$.[7] The exogenous variables include a constant term and two independently distributed random variables, one following a normal distribution with mean 3 and variance 1 and the other following a uniform distribution on the interval $[-2, 2]$. In the experiment, $\boldsymbol{\beta}_0$ is fixed at $(0.8, 0.2, 1.5)'$ and λ_0 is positive (varying from 0.9 to 0.1) in Table 1, where ρ_0 takes on a wide range of values (positive, negative, and 0.)[8] (The appendix also reports the results under negative λ_0 (varying from -0.9 to -0.1).) These configurations represent different degrees of spatial correlation in the outcome variable and the error term. The innovation term v is simulated as a zero-mean normal random variable with variance following a uniform distribution on the interval $[0.5, 4.5]$.

Insert Table 1 here.

Table 1 reports the Monte Carlo bias and root mean squared error (RMSE) from 10,000 simulations, as well as empirical rejection probability (P) of the t-test for testing the parameter equal to its true value at 5% for each parameter across the three estimation methods. Four striking observations can be made: (i) The proposed II estimator is almost unbiased in all cases. The GMM estimator is also almost unbiased in all cases (and on some occasions slightly better than the II estimator), but the GS2SLS procedure delivers substantial biases in estimating λ, ρ, and β_1 (the parameter associated with the constant term) under high degree of positive spatial correlation in the disturbance term ($\rho_0 = 0.9$), regardless of the value of λ_0. (ii) The II estimator achieves the smallest RMSE across the three estimators in the majority of all the cases considered. Under high degree of positive spatial correlation in the disturbance term, the GS2SLS method gives much larger RMSEs (relative to II and GMM) for estimating λ and ρ. This may not be surprising given the substantial biases of the GS2SLS estimator. Also, with $\rho_0 = 0.9$, the GMM estimator delivers extremely large RMSEs for estimating β_1 in spite of its small biases. This indicates that in this case there is a huge degree of uncertainty associated with the estimated intercept term from the GMM procedure. (iii) Under high degree of positive spatial correlation in the disturbance term, the GMM- and GS2SLS-based t-tests display substantial size distortions for testing λ and ρ and the GS2SLS-based t-test is also severely upward-sized for testing β_1. In contrast, the II-based t-test delivers very good finite-sample size performance in all cases. (iv) When ρ_0 is negatively large, the GMM-based t-test displays non-negligible upward size distortions for testing β_1 and β_2.

The county contiguity matrix is sparse. One may wonder about the performance of the II estimator under dense spatial weight matrices.[9] Suppose now the elements of the normalized weight matrices are of order $O(h_n^{-1})$ such that $h_n \to \infty$ and $h_n/n \to 0$ as $n \to \infty$. This corresponds to the scenario when the row and column sums of the (nonnormalized) weight matrices might diverge to infinity, as long as the number of cross-sectional units goes to infinity faster. For example, if the inverse distance measure is used in specifying the spatial weight matrix, Elhorst et. al. (2020) showed that this scenario happens when the inverse distance is raised to a positive power. With this modification, one can show that

$$\begin{pmatrix} \sqrt{n}\left(\hat{\lambda}(\rho_0) - \lambda_0 - \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n}\right) \\ \sqrt{\frac{n}{h_n}}\left(\hat{\rho}(\lambda_0,\rho_0) - \rho_0 - \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}\right) \end{pmatrix}$$

$$\xrightarrow{d} \mathrm{N}\left(\mathbf{0}, \begin{pmatrix} \lim_{n\to\infty} \frac{n\{\mathrm{tr}[\boldsymbol{\Sigma}_n\boldsymbol{E}_n\boldsymbol{\Sigma}_n(\boldsymbol{E}_n+\boldsymbol{E}_n')]+\boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{\Sigma}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\}}{[\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1})+\boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0]^2} & 0 \\ 0 & \lim_{n\to\infty} \frac{\frac{n}{h_n}\mathrm{tr}[\boldsymbol{\Sigma}_n\boldsymbol{L}_n\boldsymbol{\Sigma}_n(\boldsymbol{L}_n+\boldsymbol{L}_n')]}{[\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)]^2} \end{pmatrix}\right)$$

and Theorems 2 and 3 need to be modified accordingly. While the recentered estimator of λ_0 has the typical convergence rate $\sqrt{n}$, the recentered estimator of ρ_0 has a slower convergence rate $\sqrt{n/h_n}$. It can be shown that the resulting II estimator of ρ_0 also has the slower convergence rate of $\sqrt{n/h_n}$ and the II estimator of λ_0 and $\boldsymbol{\beta}_0$ are $\sqrt{n}$-consistent. This implies that in finite samples, one may expect poor performance of the II estimator of ρ_0 under dense spatial weight matrices.

Insert Table 2 here.

Table 2 reports results from 10,000 simulations under the circular weight matrices of Kelejian and Prucha (1999), under which each spatial unit has J neighboring units with $J/2$ neighbors "ahead" and $J/2$ neighbors "behind." The exogenous covariates $\boldsymbol{X}_n$ (and the corresponding parameter vector $\boldsymbol{\beta}_0$) and error innovations $\boldsymbol{v}_n$ follow the same experimental design as before in Table 1. The SARAR parameters (λ_0, ρ_0) are such that λ_0 is fixed at 0.4 and ρ_0 varies from 0.9 to 0.0. With a sample size of 200, the spatial weight matrices $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ $(= \boldsymbol{W}_n)$ display different degrees of density ($J = 10, 20, 100$, corresponding to 5%, 10%, and 50%, respectively, of the sample size). It can be seen that the GMM procedure produces relatively small biases in estimating λ_0 and ρ_0, but the corresponding t-test displays substantial upward size distortions. The GMM method has trouble in estimating the intercept term. For given $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$, the GS2SLS approach performs worse as the degree of spatial correlation in the error term goes up and for a given $\boldsymbol{\theta}_0$, it performs worse as $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ become denser.[10] The GS2SLS-based t-test, similar to that based on GMM, can be severely upward sized in testing the SARAR parameters, especially when $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ are dense and/or ρ_0 is large. The II procedure estimates λ_0 reasonably well across different J's, but can have serious trouble in estimating ρ_0 as J goes up. This is consistent with the statement earlier that with dense weight matrices the

II estimator (of ρ_0) may have a much slower convergence rate. The t-test from the II procedure can also be over-sized in small samples, though not as bad as the GMM and GS2SLS procedures. Additional simulation results under other parameter configurations, different degrees of density of the spatial weight matrices, and larger sample sizes are collected in the appendix.[11]

EMPIRICAL STUDIES

In this section, two empirical studies are provided. The first one is based on the exercise in Lin and Lee (2010) on county teenage pregnancy rates in 10 Upper Great Plains states in the U.S. and the second one is on the Airbnb listing prices in the city of Asheville, North Carolina in the U.S.

Teenage Pregnancy Rates

Using the data "Health and Healthcare in the United States - County and Metro Area Data" (Thomas (1999)) and the 1990 US Census (U.S. Census Bureau (1992)), Lin and Lee (2010) estimated a SAR(1) model by GMM and found strong spatial correlation among county teenage pregnancy rates. The SAR(1) model used in Lin and Lee (2010) is as follows:

$$Teen_i = \lambda \sum_{j=1}^{n} w_{ij} Teen_j + \beta_1 + Edu_i \beta_2 + Inco_i \beta_3 + FHH_i \beta_4 + Black_i \beta_5 + Phy_i \beta_6 + u_i, \quad (9)$$

where $Teen_i$ is the teenage pregnancy rate, w_{ij} is the entry from $\boldsymbol{W}_n$ (the row-normalized county contiguity matrix), Edu_i is the education service expenditure (divided by 100), $Inco_i$ is median household income (divided by 1000), FHH_i is percentage of female-headed households, $Black_i$ is proportion of black population, and Phy_i is the number of physicians per 1000 population.[12]

As pointed out by Kelejian and Prucha (1998), it is important to test the presence of possible spatial correlation in disturbances. The $\mathcal{I}^2(1)$ of Liu and Prucha (2018) applied to $Teen$ is 317.2698, yielding virtually a p-value of zero. This indicates strong cross-sectional dependence in the dependent variable. Meanwhile, the $\mathcal{I}_u^2(1)$ statistic ($\mathcal{I}^2(1)$ applied to the SAR(1) residuals) is 0.8011 with a p-value of 0.37, implying that the cross-sectional dependence in disturbances is statistically insignificant. So the results are consistent with the SAR(1) specification used in Lin and Lee (2010).

Insert Table 3 here.

Suppose one still proceeds to estimate a SARAR(1,1) model (with $\boldsymbol{M}_n = \boldsymbol{W}_n$), then one would expect that the estimated spatial autoregressive coefficient in the disturbance should be insignificant. Table 3 reports the estimated parameter values and corresponding t-statistics (absolute values in parentheses) from the GMM, GS2SLS, and II procedures, which yield comparable results. Consistent with the test statistics of Liu and Prucha (2018), the coefficient ρ is insignificant, while λ for the dependent variable is significant. The results support the findings in Hogan and Kitagawa (1985), Jencks and Mayer (1990), Case and Katz (1991), Crane (1991), Evans et al. (1992), and Lin and Lee (2010) regarding the important effect of social interaction on teenage pregnancy. The estimated parameter values of control variables are similar to those reported in Lin and Lee (2010): higher percentage of female-headed households and higher proportion of black population are associated with higher teenage pregnancy rate and factors like education expenditure, median household income and the number of physicians have the opposite effects.

Airbnb Listing Prices

The new business model of sharing economy has experienced rapid growth in recent years. In a peer-to-peer fashion, individuals rent out underused resources to other individuals in the sharing economy. Airbnb, usually described as a pioneer of the sharing economy, is an online platform that connects individuals seeking to rent accommodation assets with individuals looking for accommodations. The outburst of Airbnb has also attracted attentions from scholars and policy makers. Gutiérrez et al. (2017) and Zervas et al. (2017) studied the impact of Airbnb on the hotel industry. Lee (2016), Barron et al. (2018), and Horn and Merante (2017) investigated how Airbnb affects the housing market. Fang et al. (2016) explored the effect of Airbnb on tourism industry employment.

It is wildly acknowledged that price is one of the most critical factors in the long-term success of the accommodation sector (Hung et al. 2010). Many studies have explored the price determinants of Airbnb's shared accommodations. For example, by examining accommodation offers from 33 cities listed on Airbnb, Wang and Nicolau (2017) found that there are 5 categories of price determinants: host attributes, site and property attributes, amenities and services, rental rules, and online review ratings. Benítez-Aurioles (2018a, 2018b) explained the role of distance to city center and flexible cancellation policies in Airbnb's listing prices. Ert et al. (2016) found that the level of host trustworthiness, mainly inferred from listing photos, affects listing prices and the probability of being chosen. However, the aforementioned papers did not take into consideration of spatial correlation in Airbnb's listing prices. By using micro and aggregate data of accommodation prices listed on Airbnb in the urban area of Madrid, López et al. (2020) estimated a spatial seemingly unrelated regressions hedonic model and they found statistically significant spatial correlation.

Insert Table 4 here.

In this paper, SARAR(1,1) is applied to Airbnb accommodation log prices in Asheville, the largest city in Western North Carolina in the United States. There are in total 2247 accommodation offers in the sample.[13] The set of explanatory variables used are listed and defined in Table 4, corresponding to the 5 categories of price determinants as in Wang and Nicolau (2017). The weight matrix $\boldsymbol{W}_n$ is specified as row-normalized J-nearest neighbor weight matrix with $J = 20, 50, 100$ and $\boldsymbol{M}_n = \boldsymbol{W}_n$.

Table 5 shows the estimation results. The estimated parameter values and corresponding t-statistics (in absolute values) are quite similar across three different estimation procedures.[14] One can see that the coefficient λ is statistically significant and indicates stronger degree of spatial correlation as the number of nearest neighbors goes up. This result is consistent with López et al. (2020). In contrast to λ, while the II method indicates absence of spatial correlation in the disturbance term, the GMM and GS2SLS methods report statistically significant $\hat{\rho}$ when $J = 20$. Given the more reliable performance of the II approach as indicated in the Monte Carlo experiments (when $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ are relatively sparse), it is more reasonable to believe that there is little evidence of spatial correlation in the disturbance term. The parameter estimates of coefficients of control variables are similar to those reported in Wang and Nicolau (2017) and López et al. (2020). It is interesting to note that WiFi does not seem to affect prices, suggesting that it is perhaps taken as granted in the sharing economy of Airbnb. The number of bathrooms appears to be far more important than the number of bedrooms in property attributes, hinting that bathroom privacy is valued much more in this market. While a higher review score gives rise to a higher price tag, the number of reviews per month indicates the opposite, consistent with the phenomenon that dissatisfied customers are more likely to leave reviews, usually very critical, than happy guests.

Insert Table 5 here.

CONCLUSIONS

This paper considers the II estimation method of SARAR(1,1) model by matching the OLS estimator of the two spatial autoregressive coefficients (one in the outcome equation and the other in the error process) with its approximate analytical expectation. It is shown that the resulting II estimator is consistent, asymptotically normal, and robust to unknown heteroscedasticity. Compared with the existing estimators that rely on IV and some moment conditions associated with the error innovation term, the II estimator is found to perform better in a Monte Carlo study that uses a sparse county contiguity weight matrix. Moreover, when the degree of spatial correlation in the disturbance is high, inference procedures based on other methods can lead to severe upward size distortions, but the II-based t-test delivers very good size performance. However, when dense spatial weight matrices are employed, the estimators, including II, do not perform so well in small samples. The new estimation procedure is applied to empirical studies on teenage pregnancy rates and Airbnb accommodation prices, showing strong presence of spatial correlation in the outcome variables but little evidence of correlation in disturbances for both cases.

For future research, it is of interest to apply the II estimation method to higher-order SARAR models as in Badinger and Egger (2011), Lee and Liu (2010) and Jin and Lee (2019), among others. Again, the existing literature is largely rooted in the IV/GMM framework. The II approach aims to rely on no IV or linear and quadratic moment conditions. Another possible extension is to consider spatial panel models as in, among others, Lee and Yu (2010a, 2010b, 2010c), Baltagi et al. (2013), Elhorst (2014), and Catania and Billé (2017).

In this paper, the spatial weight matrices $\boldsymbol{M}_n$ and $\boldsymbol{W}_n$ are taken as given. In the empirical study of Airbnb accommodation prices, different weight matrices based on nearest neighbors are used and no attempt was made to decide which weight matrix specification gives the best performance. One may follow the approach of Kelejian and Piras (2011) to consider a test that compares the prediction power from a null model and that from an alternative model, where in its first step, one needs to estimate model parameters under each model specification. Another approach may be to follow Lam and Souza's (2020) LASSO strategy in selection of the weight matrices, where the LASSO objective function is based on some distance measure constructed using IV's. It would be interesting to explore testing strategies using the II estimator in the first step of Kelejian and Piras (2011) or the sample binding functions in the LASSO objective function of Lam and Souza (2020) and this is left for future research.

NOTES

[1]Two closely related papers are Liu and Yang (2015) and Breitung and Wigger (2018). They re-defined the the score function of the the log-likelihood function such that the resulting moment conditions are in fact robust to heteroscedasticity and distributional assumptions.

[2]The major difference between them is that the binding function in Kyriacou et al. (2019) comes from approximating the expectation of the ratio that defines the OLS estimator of the SAR parameter by the ratio of expectations, but in Bao et al. (2020) it is approximated such that one takes only the expectation of the numerator. In the end, the SAR parameter appears in both the numerator and denominator of the sample binding function in Kyriacou et al. (2019) and it appears only in the numerator in Bao et al. (2020). The primitive condition on the invertbility of the binding function in Kyriacou et al. (2019) then seems to be more restrictive.

[3]Kyriacou et al. (2017) used $\mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1})/\mathrm{E}(\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n)$ as the correction term for $\hat{\lambda}$ for the SAR(1) model. This makes the asymptotic variance of the recentered $\hat{\lambda}$ more complicated and it involves the kurtosis of the disturbance term under homoscedasticity.

[4]It is beyond the scope of this paper to list a set of primitive conditions to ensure the existence and uniqueness of the root for any given sample. It will depend on the structure of the data matrix, the characteristics of the weight matrices, and the parameter space. For a given sample, however, one can always plot the binding function $\boldsymbol{b}_n(\lambda, \rho)$ against (λ, ρ) to verify numerically validity of this assumption.

[5]This follows similarly from Proposition 2 of Lin and Lee (2010).

[6] While other choices are possible, in this paper, for the GMM estimator of Jin and Lee (2019), the vector of moment conditions is $(\boldsymbol{v}_n'(\boldsymbol{\theta})\boldsymbol{Q}_n, \boldsymbol{v}_n'(\boldsymbol{\theta})\boldsymbol{P}_{1n}\boldsymbol{v}_n(\boldsymbol{\theta}), \boldsymbol{v}_n'(\boldsymbol{\theta})\boldsymbol{P}_{2n}\boldsymbol{v}_n(\boldsymbol{\theta}))'$, where $\boldsymbol{Q}_n = (\boldsymbol{X}_n, \boldsymbol{W}_n\boldsymbol{X}_n^*, \boldsymbol{W}_n^2\boldsymbol{X}_n^*)$ ($\boldsymbol{X}_n^*$ denotes the part of $\boldsymbol{X}_n$ without the constant term), $\boldsymbol{P}_{1n} = \boldsymbol{W}_n$, and $\boldsymbol{P}_{2n} = \boldsymbol{W}_n^2 - \mathrm{Dg}(\boldsymbol{W}_n^2)$; for the GS2SLS estimator of Kelejian and Prucha (2010), the matrix of instrumental variables is $\boldsymbol{Q}_n$ in the first step and $(\boldsymbol{v}_n'(\boldsymbol{\theta})\boldsymbol{P}_{1n}\boldsymbol{v}_n(\boldsymbol{\theta}), \boldsymbol{v}_n'(\boldsymbol{\theta})\boldsymbol{P}_{2n}\boldsymbol{v}_n(\boldsymbol{\theta}))'$ with $\boldsymbol{P}_{1n} = \boldsymbol{M}_n$ and $\boldsymbol{P}_{2n} = \boldsymbol{M}_n'\boldsymbol{M}_n - \mathrm{Dg}(\boldsymbol{M}_n'\boldsymbol{M}_n)$ is used as the moment conditions in the second step. With such choices of the moment conditions, the GMM and GS2SLS estimators are robust to heteroscedasticity. For both, the optimal two-step GMM estimation is used. The GEL estimator in Jin and Lee (2019) is not considered in this paper, as it is much more computationally intensive and also it was shown in Jin and Lee's (2019) Monte Carlo studies that the improvement over GMM was marginal.

[7]As a referee pointed out, under a SARAR(1,1) specification, $\boldsymbol{M}_n = \boldsymbol{W}_n$ puts at risk the identification of the spatial autoregressive parameters when their true values are near zero. This is not the case in the simulation set-up though.

[8]The authors thank a referee for suggesting including negative spatial autoregressive parameters in the simulations.

[9]The authors thank the editor-in-chief and an anonymous referee for suggesting this line of discussion.

[10]It should be pointed out further that when $J = 100$, the GS2SLS fails (in terms of the optimization routine in Matlab R2020a that is used in conducting numerical estimation in this paper) more than 50% of the time, but the GMM and II rarely fail. Under $J = 10$ and 20, all the three methods have virtually zero failing rate. Table 2 reports simulation results with successful optimizations for each estimator.

[11]Observations can be made from these additional results are that as the weight matrices become denser, all the three estimators perform less reliably in small samples and that the II estimator usually performs relatively better among the three, but it may become more problematic in estimating ρ_0 accurately when each spatial unit has more neighbors.

[12]The authors are grateful to Xu Lin for providing the teenage pregnancy rate data.

[13]The sample is retrieved from a third-party website, http://insideairbnb.com/, which provides data collected from publicly available information at https://www.airbnb.com/. The sample contains 2247 accommodation offers in Asheville on March 21, 2020, including 1728 entire homes/apartments and 519 private rooms. Since only 10 shared rooms were available in Asheville on March 21, 2020, they are excluded from the sample.

[14]With the choice of $\boldsymbol{P}_{1n} = \boldsymbol{W}_n$ and $\boldsymbol{P}_{2n} = \boldsymbol{W}_n^2 - \mathrm{Dg}(\boldsymbol{W}_n^2)$ (see endnote 6), the (two-step optimal) GMM fails numerically. Instead, four quadratic moment conditions are used for the GMM estimator: $\boldsymbol{P}_{1n} = \boldsymbol{W}_n$ and $\boldsymbol{P}_{in} = \boldsymbol{W}_n^i - \mathrm{Dg}(\boldsymbol{W}_n^i)$, $i = 2, \cdots, 4$.

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Table 1: GMM, GS2SLS, and II under County Contiguity Weight Matrices ($n = 761$) and Positive λ_0

	GMM			GS2SLS			II		
$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$\lambda_0 = 0.9$	-0.010	0.052	10.9%	0.004	0.049	10.0%	-0.003	0.035	7.4%
$\rho_0 = 0.6$	0.000	0.097	8.3%	-0.029	0.102	6.5%	-0.008	0.074	5.0%
$\beta_{10} = 0.8$	0.208	1.155	10.4%	-0.093	1.082	9.6%	0.071	0.790	6.8%
$\beta_{20} = 0.2$	0.001	0.055	5.4%	0.001	0.064	5.1%	0.001	0.055	5.2%
$\beta_{30} = 1.5$	-0.001	0.050	5.2%	-0.009	0.054	5.3%	0.000	0.050	4.9%
$\lambda_0 = 0.9$	-0.004	0.032	9.1%	0.002	0.032	6.4%	-0.002	0.024	5.3%
$\rho_0 = 0.3$	-0.003	0.094	7.3%	-0.018	0.098	6.0%	-0.007	0.080	5.3%
$\beta_{10} = 0.8$	0.080	0.716	8.4%	-0.026	0.715	5.9%	0.048	0.536	4.7%
$\beta_{20} = 0.2$	0.000	0.059	5.9%	-0.001	0.059	5.4%	0.000	0.058	5.1%
$\beta_{30} = 1.5$	0.000	0.052	5.4%	-0.004	0.052	5.1%	0.000	0.052	5.1%
$\lambda_0 = 0.9$	-0.002	0.024	10.0%	0.001	0.026	6.2%	-0.001	0.019	5.8%
$\rho_0 = 0.0$	-0.004	0.092	7.3%	-0.014	0.098	6.7%	-0.007	0.081	5.1%
$\beta_{10} = 0.8$	0.046	0.557	9.8%	-0.010	0.567	5.5%	0.033	0.435	5.4%
$\beta_{20} = 0.2$	-0.001	0.060	6.7%	-0.001	0.058	5.3%	-0.001	0.058	5.4%
$\beta_{30} = 1.5$	0.001	0.054	5.7%	-0.001	0.054	5.2%	0.001	0.053	5.5%
$\lambda_0 = 0.9$	-0.001	0.020	9.7%	0.000	0.022	5.4%	-0.001	0.016	5.2%
$\rho_0 = -0.3$	-0.004	0.091	8.2%	-0.010	0.098	8.2%	-0.006	0.080	5.5%
$\beta_{10} = 0.8$	0.031	0.475	10.2%	0.003	0.482	5.2%	0.030	0.370	5.1%
$\beta_{20} = 0.2$	-0.001	0.063	7.9%	-0.001	0.059	5.1%	-0.001	0.057	5.3%
$\beta_{30} = 1.5$	0.000	0.056	6.0%	-0.001	0.056	5.1%	0.000	0.054	5.3%
$\lambda_0 = 0.9$	-0.001	0.018	11.0%	0.000	0.020	5.1%	-0.001	0.014	5.4%
$\rho_0 = -0.6$	-0.002	0.086	8.5%	-0.005	0.094	9.0%	-0.004	0.075	5.5%
$\beta_{10} = 0.8$	0.022	0.438	12.0%	0.007	0.446	5.1%	0.023	0.330	5.3%
$\beta_{20} = 0.2$	0.000	0.065	9.6%	0.000	0.062	5.0%	0.000	0.056	5.1%
$\beta_{30} = 1.5$	-0.001	0.059	6.7%	-0.001	0.062	5.1%	0.000	0.054	5.2%
$\lambda_0 = 0.4$	0.004	0.111	11.3%	0.163	0.253	34.0%	0.025	0.089	4.9%
$\rho_0 = 0.9$	-0.010	0.055	13.1%	-0.120	0.188	24.0%	-0.018	0.044	3.4%
$\beta_{10} = 0.8$	0.006	6.439	8.5%	-0.560	1.044	29.6%	-0.094	0.671	7.3%
$\beta_{20} = 0.2$	0.001	0.054	5.2%	-0.003	0.081	5.6%	0.001	0.054	5.0%
$\beta_{30} = 1.5$	-0.002	0.058	7.0%	-0.027	0.076	6.4%	0.005	0.054	5.0%
$\lambda_0 = 0.4$	-0.004	0.098	8.5%	0.016	0.111	8.7%	0.012	0.092	7.9%
$\rho_0 = 0.6$	-0.009	0.089	8.6%	-0.030	0.103	8.4%	-0.024	0.089	7.0%
$\beta_{10} = 0.8$	0.020	0.436	7.5%	-0.051	0.477	7.3%	-0.041	0.409	6.6%
$\beta_{20} = 0.2$	-0.001	0.056	5.1%	-0.001	0.064	5.0%	0.000	0.055	4.9%
$\beta_{30} = 1.5$	-0.003	0.053	5.4%	-0.007	0.055	5.1%	0.001	0.052	4.9%
$\lambda_0 = 0.4$	-0.004	0.067	6.8%	0.003	0.065	5.6%	0.002	0.064	5.9%
$\rho_0 = 0.0$	-0.005	0.105	7.1%	-0.015	0.105	5.3%	-0.013	0.101	6.1%
$\beta_{10} = 0.8$	0.014	0.317	7.6%	-0.009	0.294	5.7%	-0.008	0.289	5.8%
$\beta_{20} = 0.2$	0.000	0.060	6.6%	0.000	0.058	5.2%	0.000	0.058	5.3%
$\beta_{30} = 1.5$	-0.002	0.052	5.1%	-0.002	0.052	5.0%	-0.002	0.052	5.2%
$\lambda_0 = 0.4$	-0.003	0.054	7.2%	-0.002	0.059	4.9%	0.000	0.050	5.6%
$\rho_0 = -0.6$	-0.003	0.098	7.5%	-0.005	0.105	6.4%	-0.007	0.090	5.2%
$\beta_{10} = 0.8$	0.016	0.278	10.4%	0.011	0.265	4.9%	0.004	0.231	4.6%
$\beta_{20} = 0.2$	-0.001	0.065	9.6%	-0.001	0.062	4.9%	-0.001	0.056	5.4%
$\beta_{30} = 1.5$	0.000	0.057	5.8%	-0.001	0.059	5.0%	-0.002	0.056	5.1%
$\lambda_0 = 0.4$	-0.005	0.049	5.8%	-0.005	0.064	5.6%	-0.005	0.044	3.7%
$\rho_0 = -0.9$	0.004	0.080	2.9%	0.013	0.094	3.8%	0.011	0.067	1.7%
$\beta_{10} = 0.8$	0.019	0.269	11.9%	0.016	0.283	5.3%	0.012	0.212	4.3%
$\beta_{20} = 0.2$	0.000	0.067	12.0%	0.000	0.069	5.4%	0.001	0.054	5.3%
$\beta_{30} = 1.5$	0.002	0.060	6.0%	-0.001	0.067	5.5%	0.003	0.058	5.0%
$\lambda_0 = 0.1$	0.005	0.107	10.5%	0.189	0.291	30.7%	0.020	0.084	4.3%
$\rho_0 = 0.9$	-0.007	0.042	11.8%	-0.101	0.159	30.4%	-0.012	0.032	3.6%
$\beta_{10} = 0.8$	0.004	3.760	7.0%	-0.430	0.895	25.1%	-0.037	0.621	6.5%
$\beta_{20} = 0.2$	0.000	0.054	5.4%	-0.001	0.086	6.4%	0.000	0.054	5.2%
$\beta_{30} = 1.5$	0.000	0.061	7.9%	-0.011	0.080	5.5%	0.006	0.055	5.0%
$\lambda_0 = 0.1$	0.001	0.105	9.0%	0.022	0.122	8.0%	0.014	0.097	6.1%
$\rho_0 = 0.6$	-0.011	0.084	8.8%	-0.028	0.098	9.1%	-0.020	0.080	6.2%
$\beta_{10} = 0.8$	-0.002	0.353	7.2%	-0.048	0.392	6.0%	-0.036	0.330	5.4%
$\beta_{20} = 0.2$	0.000	0.055	5.1%	-0.001	0.064	4.6%	0.000	0.055	4.8%
$\beta_{30} = 1.5$	-0.002	0.056	6.0%	-0.005	0.058	4.7%	0.002	0.055	5.1%
$\lambda_0 = 0.1$	-0.003	0.081	6.5%	0.003	0.078	5.4%	0.004	0.078	6.0%
$\rho_0 = 0.0$	-0.007	0.105	6.5%	-0.014	0.103	4.7%	-0.014	0.103	6.2%
$\beta_{10} = 0.8$	0.008	0.285	7.3%	-0.006	0.262	5.4%	-0.008	0.263	5.6%
$\beta_{20} = 0.2$	0.000	0.061	6.9%	0.000	0.058	5.3%	0.000	0.058	5.6%
$\beta_{30} = 1.5$	-0.001	0.051	5.1%	-0.001	0.051	4.9%	-0.001	0.051	4.9%
$\lambda_0 = 0.1$	-0.003	0.068	6.7%	-0.002	0.075	5.0%	0.003	0.065	5.8%
$\rho_0 = -0.6$	-0.005	0.103	7.2%	-0.005	0.109	5.8%	-0.010	0.096	5.9%
$\beta_{10} = 0.8$	0.008	0.262	10.3%	0.006	0.247	5.1%	-0.004	0.220	5.2%
$\beta_{20} = 0.2$	-0.001	0.065	10.1%	0.000	0.063	5.1%	0.000	0.056	5.4%
$\beta_{30} = 1.5$	0.000	0.055	5.6%	-0.001	0.056	4.9%	-0.002	0.055	5.3%
$\lambda_0 = 0.1$	-0.008	0.063	5.1%	-0.008	0.086	6.2%	-0.008	0.058	3.7%
$\rho_0 = -0.9$	0.005	0.085	2.9%	0.018	0.100	3.7%	0.014	0.072	1.9%
$\beta_{10} = 0.8$	0.020	0.255	11.6%	0.023	0.269	5.6%	0.016	0.201	4.4%
$\beta_{20} = 0.2$	-0.001	0.068	11.9%	-0.001	0.069	5.5%	0.001	0.054	5.3%
$\beta_{30} = 1.5$	0.002	0.058	5.6%	-0.002	0.063	5.3%	0.003	0.056	4.6%

Table 2: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 200$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 10$	$\lambda_0 = 0.4$	-0.021	0.266	15.0%	0.487	0.552	68.8%	0.177	0.268	21.7%
	$\rho_0 = 0.9$	-0.033	0.141	24.7%	-0.401	0.496	44.1%	-0.134	0.197	8.8%
	$\beta_{10} = 0.8$	1.535	144.105	10.2%	-1.136	1.456	52.8%	-0.429	1.140	16.2%
	$\beta_{20} = 0.2$	-0.001	0.108	6.0%	-0.001	0.120	4.5%	0.003	0.110	4.6%
	$\beta_{30} = 1.5$	-0.010	0.105	7.0%	-0.010	0.108	5.1%	0.022	0.101	4.6%
	$\lambda_0 = 0.4$	-0.048	0.283	18.4%	0.207	0.341	32.8%	0.072	0.215	19.6%
	$\rho_0 = 0.7$	-0.030	0.208	21.3%	-0.218	0.350	20.8%	-0.126	0.222	10.1%
	$\beta_{10} = 0.8$	-0.776	114.976	8.2%	-0.478	0.927	21.0%	-0.173	0.701	10.8%
	$\beta_{20} = 0.2$	-0.002	0.111	6.9%	-0.002	0.118	5.1%	0.000	0.112	5.4%
	$\beta_{30} = 1.5$	-0.012	0.104	6.9%	-0.014	0.103	4.5%	0.006	0.097	4.5%
	$\lambda_0 = 0.4$	-0.050	0.251	15.0%	0.038	0.195	12.5%	0.008	0.182	12.1%
	$\rho_0 = 0.3$	-0.017	0.278	15.9%	-0.074	0.279	9.8%	-0.101	0.257	9.4%
	$\beta_{10} = 0.8$	1.321	66.086	8.5%	-0.077	0.593	6.6%	-0.007	0.573	7.2%
	$\beta_{20} = 0.2$	-0.004	0.115	7.6%	-0.004	0.115	5.8%	-0.003	0.114	6.5%
	$\beta_{30} = 1.5$	-0.011	0.102	6.8%	-0.010	0.099	5.3%	-0.005	0.098	5.2%
	$\lambda_0 = 0.4$	-0.042	0.224	12.7%	0.017	0.169	9.1%	-0.005	0.165	10.5%
	$\rho_0 = 0.1$	-0.013	0.292	12.9%	-0.042	0.284	8.2%	-0.088	0.263	9.3%
	$\beta_{10} = 0.8$	-0.145	54.165	8.2%	-0.039	0.541	5.7%	0.012	0.540	6.6%
	$\beta_{20} = 0.2$	0.000	0.117	7.4%	-0.001	0.114	5.6%	-0.001	0.115	6.3%
	$\beta_{30} = 1.5$	-0.008	0.102	7.0%	-0.007	0.099	5.5%	-0.005	0.099	5.3%
	$\lambda_0 = 0.4$	-0.037	0.213	12.0%	0.015	0.156	8.7%	-0.005	0.155	10.3%
	$\rho_0 = 0.0$	-0.013	0.299	11.9%	-0.038	0.279	7.5%	-0.083	0.268	9.2%
	$\beta_{10} = 0.8$	-0.328	38.903	8.2%	-0.030	0.511	5.7%	0.015	0.512	6.5%
	$\beta_{20} = 0.2$	-0.002	0.115	6.5%	-0.002	0.112	5.3%	-0.002	0.113	5.8%
	$\beta_{30} = 1.5$	-0.010	0.102	6.8%	-0.008	0.099	5.5%	-0.007	0.099	5.5%
$J = 20$	$\lambda_0 = 0.4$	-0.027	0.348	16.9%	0.553	0.605	72.0%	0.234	0.319	34.1%
	$\rho_0 = 0.9$	-0.056	0.213	34.8%	-0.504	0.622	42.8%	-0.226	0.310	13.7%
	$\beta_{10} = 0.8$	-6.175	323.532	12.0%	-1.300	1.543	59.6%	-0.547	1.155	25.1%
	$\beta_{20} = 0.2$	0.002	0.111	6.2%	0.002	0.114	4.5%	0.004	0.111	4.6%
	$\beta_{30} = 1.5$	-0.007	0.103	7.0%	-0.001	0.102	5.2%	0.014	0.101	4.5%
	$\lambda_0 = 0.4$	-0.083	0.398	22.0%	0.341	0.478	42.6%	0.099	0.263	25.1%
	$\rho_0 = 0.7$	-0.042	0.307	30.3%	-0.343	0.522	27.0%	-0.218	0.341	12.4%
	$\beta_{10} = 0.8$	-0.219	311.057	10.9%	-0.794	1.200	32.4%	-0.228	0.795	15.1%
	$\beta_{20} = 0.2$	-0.002	0.111	6.4%	-0.001	0.113	4.7%	0.000	0.111	4.9%
	$\beta_{30} = 1.5$	-0.013	0.102	6.6%	-0.007	0.099	4.4%	0.001	0.097	4.4%
	$\lambda_0 = 0.4$	-0.086	0.376	19.4%	0.101	0.290	16.9%	0.005	0.244	15.0%
	$\rho_0 = 0.3$	-0.021	0.409	21.7%	-0.144	0.440	14.9%	-0.185	0.375	10.8%
	$\beta_{10} = 0.8$	-1.110	236.415	10.4%	-0.230	0.776	10.4%	-0.003	0.692	9.1%
	$\beta_{20} = 0.2$	-0.003	0.115	7.0%	-0.003	0.113	5.4%	-0.002	0.114	5.9%
	$\beta_{30} = 1.5$	-0.011	0.102	6.9%	-0.007	0.099	4.9%	-0.005	0.099	5.3%
	$\lambda_0 = 0.4$	-0.084	0.356	17.4%	0.051	0.249	11.2%	-0.015	0.233	12.7%
	$\rho_0 = 0.1$	-0.005	0.436	18.6%	-0.077	0.423	10.8%	-0.169	0.384	10.3%
	$\beta_{10} = 0.8$	0.749	204.528	9.5%	-0.113	0.692	6.8%	0.040	0.667	7.7%
	$\beta_{20} = 0.2$	-0.002	0.115	6.6%	-0.001	0.113	5.2%	-0.001	0.114	5.9%
	$\beta_{30} = 1.5$	-0.008	0.103	7.2%	-0.004	0.099	5.4%	-0.004	0.099	5.6%
	$\lambda_0 = 0.4$	-0.078	0.336	16.0%	0.029	0.227	8.4%	-0.022	0.230	11.5%
	$\rho_0 = 0.0$	-0.003	0.449	17.4%	-0.044	0.409	8.8%	-0.156	0.389	10.4%
	$\beta_{10} = 0.8$	0.159	196.797	9.0%	-0.064	0.638	5.4%	0.054	0.652	7.0%
	$\beta_{20} = 0.2$	-0.002	0.115	7.2%	-0.001	0.113	5.5%	-0.002	0.114	6.1%
	$\beta_{30} = 1.5$	-0.009	0.101	6.6%	-0.004	0.098	5.5%	-0.005	0.099	5.7%
$J = 100$	$\lambda_0 = 0.4$	0.143	0.601	18.8%	0.760	1.072	43.7%	0.301	0.531	16.4%
	$\rho_0 = 0.9$	-0.109	0.491	50.1%	-0.770	1.018	31.5%	-1.078	1.216	24.0%
	$\beta_{10} = 0.8$	10.883	606.612	13.9%	-1.770	2.936	40.9%	-0.709	1.624	28.9%
	$\beta_{20} = 0.2$	-0.001	0.114	6.4%	0.000	0.115	5.6%	-0.001	0.114	5.7%
	$\beta_{30} = 1.5$	0.002	0.100	6.6%	0.009	0.099	5.5%	0.004	0.099	5.8%
	$\lambda_0 = 0.4$	0.044	0.641	22.9%	0.603	1.014	34.1%	0.193	0.510	14.4%
	$\rho_0 = 0.7$	-0.034	0.601	48.7%	-0.638	0.939	25.0%	-1.061	1.193	22.7%
	$\beta_{10} = 0.8$	1.734	586.217	17.7%	-1.423	2.454	32.0%	-0.451	1.307	14.3%
	$\beta_{20} = 0.2$	0.001	0.114	6.5%	0.001	0.113	5.8%	0.001	0.114	6.0%
	$\beta_{30} = 1.5$	0.002	0.099	6.0%	0.010	0.099	5.7%	0.005	0.098	5.5%
	$\lambda_0 = 0.4$	-0.062	0.691	27.3%	0.366	0.974	18.1%	0.003	0.512	10.8%
	$\rho_0 = 0.3$	0.066	0.796	43.2%	-0.260	0.755	22.3%	-0.863	0.998	15.3%
	$\beta_{10} = 0.8$	9.369	540.922	22.3%	-0.862	2.304	16.4%	-0.014	1.254	9.3%
	$\beta_{20} = 0.2$	-0.001	0.114	6.5%	0.000	0.113	5.4%	0.000	0.114	5.3%
	$\beta_{30} = 1.5$	-0.001	0.099	6.4%	0.007	0.099	5.4%	0.002	0.098	5.5%
	$\lambda_0 = 0.4$	-0.070	0.690	29.0%	0.229	0.926	12.3%	-0.066	0.520	10.5%
	$\rho_0 = 0.1$	0.106	0.847	38.1%	-0.023	0.686	23.8%	-0.741	0.879	10.6%
	$\beta_{10} = 0.8$	6.651	503.107	23.5%	-0.526	2.228	11.2%	0.160	1.276	8.8%
	$\beta_{20} = 0.2$	-0.002	0.116	7.1%	-0.002	0.116	6.4%	-0.002	0.115	6.0%
	$\beta_{30} = 1.5$	-0.003	0.098	6.2%	0.004	0.097	4.9%	-0.001	0.097	5.2%
	$\lambda_0 = 0.4$	-0.098	0.701	27.5%	0.120	0.969	9.3%	-0.114	0.531	10.1%
	$\rho_0 = 0.0$	0.160	0.861	37.4%	0.114	0.687	25.3%	-0.668	0.808	8.3%
	$\beta_{10} = 0.8$	0.939	492.668	22.8%	-0.280	2.337	8.1%	0.261	1.300	8.7%
	$\beta_{20} = 0.2$	0.001	0.115	6.7%	-0.002	0.115	5.9%	0.001	0.114	5.7%
	$\beta_{30} = 1.5$	-0.004	0.098	6.2%	0.001	0.097	4.4%	-0.002	0.098	5.2%

Table 3: SARAR(1,1) Fitted to County Teenage Pregnancy Rates

	λ	ρ	Constant	Edu	Inco	FHH	Black	Phy
GMM	0.4792	-0.1156	7.6036	-0.0107	-0.2409	0.6608	0.1495	-0.1877
	(7.5052)	(1.0506)	(5.5285)	(2.4390)	(7.0172)	(7.4832)	(3.1224)	(1.1137)
GS2SLS	0.4201	-0.0947	7.893	-0.0103	-0.2368	0.7443	0.144	-0.3688
	(5.6651)	(0.7769)	(5.6188)	(2.3312)	(6.5599)	(7.2083)	(2.6241)	(2.0594)
II	0.4688	-0.1774	6.9261	-0.0115	-0.2122	0.7216	0.1443	-0.3908
	(3.1151)	(0.6863)	(2.5613)	(2.3572)	(4.1099)	(7.5515)	(2.6486)	(2.1516)

Table 4: Explanatory Variables of Airbnb Prices

Variable	Mean	Std Dev	Definition
Superhost	0.7178	0.4501	Host is experiened (1) or not (0)
Host Count	10.3053	55.6603	Number of accommodation rentals listed by host
Entire Home	0.7690	0.4216	Entire home/apartment (1) or not (0)
Accommodates	4.0908	2.6398	Number of people that can be accommodated
Bedrooms	1.5928	1.2728	Number of bedrooms
Bathrooms	1.3605	0.7376	Number of bathrooms
Dist-Center	5.3845	4.0990	Distance to city center
Free Parking	0.9675	0.1773	Offer free parking (1) or not (0)
WiFi	0.9866	0.1148	Offer WiFi (1) or not (0)
TV	0.8247	0.3803	Offer TV (1) or not (0)
Breakfast	0.1397	0.3468	Offer breakfast (1) or not (0)
Instant Bookable	0.6306	0.4827	Offer instant booking (1) or not (0)
Min-Nights	4.2016	19.8402	Minimum number of nights
Reviews/month	3.0792	2.4404	Number of reviews per month
Review Score	97.6756	3.4261	Overall review scores

Table 5: SARAR(1,1) Fitted to Airbnb log(Prices) in Asheville

	$J = 20$			$J = 50$			$J = 100$		
	GMM	GS2SLS	II	GMM	GS2SLS	II	GMM	GS2SLS	II
λ	0.3315	0.4697	0.3907	0.5037	0.5246	0.4541	0.4651	0.5695	0.5342
	(6.5213)	(10.4570)	(6.7420)	(17.3028)	(18.1473)	(5.6400)	(12.7554)	(16.8049)	(10.4277)
ρ	0.4578	0.2910	0.2729	-0.1774	-0.3148	0.1456	-0.2477	-0.2109	0.0247
	(5.7353)	(2.6997)	(1.4687)	(1.1056)	(1.2174)	(0.4323)	(1.2110)	(0.7462)	(0.0733)
Constant	1.1962	0.6110	0.9528	0.4048	0.2146	0.5330	1.3686	0.1397	0.3241
	(2.7640)	(1.4846)	(2.1822)	(1.1200)	(0.5445)	(1.0216)	(3.9427)	(0.3561)	(0.7958)
Superhost	-0.0148	0.0110	0.0115	-0.0099	0.0131	0.0142	0.0015	0.0152	0.0156
	(0.7107)	(0.4429)	(0.4587)	(0.4749)	(0.5110)	(0.5598)	(0.0735)	(0.5982)	(0.6110)
HostCount	0.0012	0.0014	0.0014	0.0015	0.0014	0.0014	0.0012	0.0014	0.0014
	(8.0270)	(5.9492)	(5.8199)	(6.9658)	(5.4783)	(5.5286)	(6.8785)	(5.4004)	(5.4661)
EntireHome	0.3527	0.3325	0.3417	0.3351	0.3351	0.3399	0.3323	0.3352	0.3363
	(16.9585)	(14.4085)	(14.9722)	(15.7112)	(14.7637)	(14.8598)	(15.6642)	(14.7971)	(14.7870)
Accommodates	0.0677	0.0792	0.0813	0.0950	0.0835	0.0844	0.1098	0.0846	0.0847
	(5.1678)	(6.8516)	(6.9728)	(13.9441)	(7.4875)	(7.5117)	(16.5536)	(7.5084)	(7.4518)
Bedrooms	0.0643	0.0089	0.0105	-0.0365	0.0025	0.0027	-0.0650	0.0047	0.0044
	(1.6583)	(0.2818)	(0.3246)	(2.5258)	(0.0835)	(0.0904)	(5.3425)	(0.1574)	(0.1439)
Bathrooms	0.1841	0.1890	0.1885	0.2022	0.1915	0.1929	0.1908	0.1894	0.1909
	(8.6421)	(8.3507)	(8.4181)	(9.0445)	(8.3139)	(8.3567)	(8.3666)	(8.1846)	(8.2655)
Dis-Center	-0.0274	-0.0226	-0.0229	-0.0246	-0.0241	-0.0244	-0.0216	-0.0223	-0.0225
	(7.7647)	(6.9425)	(7.3729)	(13.5064)	(12.9272)	(8.9282)	(12.3243)	(11.0878)	(9.1162)
FreeParking	-0.1439	-0.1067	-0.1046	-0.1707	-0.0846	-0.0992	-0.2145	-0.0885	-0.1010
	(2.2974)	(1.6924)	(1.7421)	(3.3181)	(1.4232)	(1.6311)	(3.9404)	(1.4435)	(1.7247)
WiFi	-0.0029	0.0023	-0.0065	0.0291	-0.0135	-0.0159	-0.1071	-0.0332	-0.0330
	(0.0495)	(0.0354)	(0.1027)	(0.4331)	(0.1966)	(0.2348)	(1.4807)	(0.4713)	(0.4705)
TV	0.1612	0.1677	0.1711	0.1623	0.1751	0.1785	0.1660	0.1805	0.1836
	(7.6246)	(7.2502)	(7.5637)	(7.5196)	(7.6285)	(7.4725)	(7.5411)	(7.7063)	(7.6158)
Breakfast	0.0801	0.0717	0.0679	0.1004	0.0781	0.0792	0.1009	0.0745	0.0760
	(3.4023)	(2.7937)	(2.6859)	(4.2895)	(3.0939)	(3.1125)	(4.1522)	(2.9221)	(2.9975)
InstantBookable	0.0698	0.0818	0.0824	0.0769	0.0818	0.0830	0.0718	0.0808	0.0820
	(4.1445)	(4.3729)	(4.5153)	(4.5534)	(4.5129)	(4.5002)	(4.2395)	(4.4491)	(4.4633)
Min-Nights	-0.0008	-0.0013	-0.0013	-0.0016	-0.0013	-0.0014	-0.0015	-0.0014	-0.0014
	(1.2586)	(1.2701)	(1.3264)	(1.4698)	(1.3267)	(1.4091)	(1.5318)	(1.3911)	(1.4267)
Reviews/month	-0.0377	-0.0404	-0.0406	-0.0408	-0.0412	-0.0415	-0.0419	-0.0408	-0.0411
	(9.8417)	(8.9226)	(9.2693)	(10.6326)	(9.1167)	(9.2057)	(11.1046)	(8.9544)	(8.9774)
ReviewScore	0.0133	0.0122	0.0124	0.0132	0.0133	0.0136	0.0068	0.0120	0.0119
	(3.8718)	(3.5025)	(3.5493)	(3.9488)	(3.7532)	(3.8195)	(2.3234)	(3.4507)	(3.4484)

Estimating a Spatial Autoregressive Model with Autoregressive Disturbances Based on the Indirect Inference Principle

Appendix A: Lemmas and Proofs

This appendix first collects several lemmas that are useful for deriving the main results. $\odot$ denotes matrix Hadamard product operator and $\text{dg}(\boldsymbol{A}_n)$ is a column vector that collects in order the diagonal elements of the square matrix $\boldsymbol{A}_n$.

Lemma 1. *Suppose $\{\boldsymbol{A}_n\}$ is a sequence of matrices with row and column sums that are bounded uniformly in absolute value. Let $\{\boldsymbol{b}_n\}$ be a sequence of constants with uniformly bounded elements and $\sup n^{-1}\sum_{i=1}^n |b_{i,n}|^{2+\eta} < \infty$ for some $\eta > 0$. For the sequence $\{\boldsymbol{v}_n\}$ that satisfies Assumption 3, let $Q_n = \boldsymbol{b}_n'\boldsymbol{v}_n + \boldsymbol{v}_n'\boldsymbol{A}_n\boldsymbol{v}_n$. Then*

$$\frac{Q_n - \text{E}(Q_n)}{\sqrt{\text{Var}(Q_n)}} \xrightarrow{d} \text{N}(0,1).$$

Proof. The proof follows closely Kelejian and Prucha (2001) and Lee (2002, 2004), with slight modification to allow heterogeneity in $\{v_{i,n}\}$. ■

Lemma 2. *If $\{\boldsymbol{A}_n\}$ and $\{\boldsymbol{B}_n\}$ are sequences of matrices with row and column sums that are bounded uniformly in absolute value, then $\{\boldsymbol{A}_n + \boldsymbol{B}_n\}$ and $\{\boldsymbol{A}_n\boldsymbol{B}_n\}$ are also are bounded uniformly in absolute value in row and column sums.*

Proof. See Lee (2002, 2004). ■

Lemma 3. *For the sequence $\{\boldsymbol{v}_n\}$ with elements following Assumption 3, let $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ be nonrandom, then*

$$\begin{aligned}
&\text{E}(\boldsymbol{v}_n'\boldsymbol{A}_n\boldsymbol{v}_n) = \text{tr}(\boldsymbol{\Sigma}_n\boldsymbol{A}_n),\\
&\text{E}(\boldsymbol{v}_n\boldsymbol{u}_n'\boldsymbol{A}_n\boldsymbol{v}_n) = \text{dg}(\boldsymbol{\Sigma}_n^{(3)} \odot \boldsymbol{A}_n),\\
&\text{E}(\boldsymbol{v}_n'\boldsymbol{A}_n\boldsymbol{v}_n\boldsymbol{v}_n'\boldsymbol{B}_n\boldsymbol{v}_n) = \text{tr}(\boldsymbol{\Sigma}_n^{(4)} \odot \boldsymbol{A}_n \odot \boldsymbol{B}_n) + \text{tr}(\boldsymbol{\Sigma}_n\boldsymbol{A}_n)\text{tr}(\boldsymbol{\Sigma}_n\boldsymbol{B}_n)\\
&\quad + \text{tr}[\boldsymbol{\Sigma}_n\boldsymbol{A}_n\boldsymbol{\Sigma}_n(\boldsymbol{B}_n + \boldsymbol{B}_n')],
\end{aligned}$$

where $\boldsymbol{\Sigma}_n^{(3)} = \text{Dg}(\text{E}(v_{1,n}^3), ..., \text{E}(v_{n,n}^3))$, and $\boldsymbol{\Sigma}_n^{(4)} = \text{Dg}(\text{E}(v_{1,n}^4) - 3\sigma_{1,n}^4, ..., \text{E}(v_{n,n}^4) - 3\sigma_{n,n}^4)$.

Proof. See Appendix A.7 of Ullah (2004). ■

Lemma 4. *Let $\tilde{\boldsymbol{\beta}}_n = (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$, $\tilde{\boldsymbol{v}}_n = \boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{R}_n\boldsymbol{X}_n\tilde{\boldsymbol{\beta}}_n$, and $\tilde{\boldsymbol{u}}_n = \boldsymbol{R}_n^{-1}\tilde{\boldsymbol{v}}_n$. Then under Assumptions 1–3, 5, $\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n\tilde{\boldsymbol{v}}_n - \boldsymbol{u}_n'\boldsymbol{M}_n\boldsymbol{v}_n = O_p(1)$, $\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n - \boldsymbol{u}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{u}_n = O_p(1)$, $\tilde{\boldsymbol{v}}_n'\boldsymbol{D}_n\tilde{\boldsymbol{v}}_n - \boldsymbol{v}_n'\boldsymbol{D}_n\boldsymbol{v}_n = O_p(1)$, and $\tilde{\boldsymbol{v}}_n'\boldsymbol{L}_n\tilde{\boldsymbol{v}}_n - \boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n = O_p(1)$. Also, $[\boldsymbol{v}_n'\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{v}_n - \text{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)]/\text{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n) = O_p(n^{-1/2})$.*

Proof. By substitution, one has $\tilde{\boldsymbol{u}}_n' \boldsymbol{M}_n \tilde{\boldsymbol{v}}_n - \boldsymbol{u}_n' \boldsymbol{M}_n \boldsymbol{v}_n = \tilde{\boldsymbol{u}}_n' \boldsymbol{M}_n \boldsymbol{R}_n \tilde{\boldsymbol{u}}_n - \boldsymbol{u}_n' \boldsymbol{M}_n \boldsymbol{R}_n \boldsymbol{u}_n = (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n)' \boldsymbol{X}_n' \boldsymbol{M}_n \boldsymbol{R}_n \boldsymbol{X}_n (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n) - 2(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n) \boldsymbol{X}_n' \boldsymbol{M}_n \boldsymbol{R}_n \boldsymbol{u}_n$. Note that $\boldsymbol{X}_n' \boldsymbol{M}_n \boldsymbol{R}_n \boldsymbol{X}_n = O(n)$ and $\mathrm{Var}(\boldsymbol{X}_n' \boldsymbol{M}_n \boldsymbol{R}_n \boldsymbol{u}_n) = \mathrm{Var}(\boldsymbol{X}_n' \boldsymbol{M}_n \boldsymbol{v}_n) = \boldsymbol{X}_n' \boldsymbol{M}_n \boldsymbol{\Sigma}_n \boldsymbol{M}_n \boldsymbol{X}_n = O_p(n)$ (by using Lemma 2). Then, in view of $(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0) = O_p(n^{-1/2})$, one can immediately see that $\tilde{\boldsymbol{u}}_n' \boldsymbol{M}_n \tilde{\boldsymbol{v}}_n - \boldsymbol{u}_n' \boldsymbol{M}_n \boldsymbol{v}_n = O_p(1)$. Other proofs are similar. The result on $[\boldsymbol{v}_n' \boldsymbol{F}_n' \boldsymbol{F}_n \boldsymbol{v}_n - \mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{F}_n' \boldsymbol{F}_n)]/\mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{F}_n' \boldsymbol{F}_n) = [\boldsymbol{v}_n' \boldsymbol{F}_n' \boldsymbol{F}_n \boldsymbol{v}_n - \mathrm{E}(\boldsymbol{v}_n' \boldsymbol{F}_n' \boldsymbol{F}_n \boldsymbol{v}_n)]/\mathrm{E}(\boldsymbol{v}_n' \boldsymbol{F}_n' \boldsymbol{F}_n \boldsymbol{v}_n)$ follows directly from Lemmas 2 and 3. ■

Lemma 5. *Under Assumptions 1–3, 5,* $\mathrm{E}(r_n) = O(n)$, $\mathrm{E}(d_n) = O(n)$, $\mathrm{Var}(r_n) = O(n)$, $\mathrm{Var}(d_n) = O(n)$, $[r_n - \mathrm{E}(r_n)]/\mathrm{E}(r_n) = O_p(n^{-1/2})$, *and* $[d_n - \mathrm{E}(d_n)]/\mathrm{E}(d_n) = O_p(n^{-1/2})$, *where* $r_n = \boldsymbol{v}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \boldsymbol{v}_n + \boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{v}_n$ *and* $d_n = \boldsymbol{v}_n' \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \boldsymbol{v}_n + 2\boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \boldsymbol{v}_n + \boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{X}_n \boldsymbol{\beta}_0$.

Proof. Lemma 3 gives $\mathrm{E}(r_n) = \mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1})$. From Lemma 2, $\boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}$ has row and column sums bounded uniformly in absolute value and it follows that $\mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}) = O(n)$. Similarly, $\mathrm{E}(d_n) = \mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}) + \boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{X}_n \boldsymbol{\beta}_0 = O(n)$. As for the variances, using Lemma 3, one has

$$\begin{aligned}
\mathrm{Var}(r_n) = {} & \mathrm{tr}(\boldsymbol{\Sigma}_n^{(4)} \odot \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \odot \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}) \\
& + \mathrm{tr}[\boldsymbol{\Sigma}_n \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \boldsymbol{\Sigma}_n (\boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} + \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n)] \\
& + \boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{\Sigma}_n \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{X}_n \boldsymbol{\beta}_0 \\
& + 2\boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \mathrm{dg}(\boldsymbol{\Sigma}_n^{(3)} \odot \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1})
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(d_n) = {} & \mathrm{tr}(\boldsymbol{\Sigma}_n^{(4)} \odot \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \odot \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}) \\
& + 2\mathrm{tr}(\boldsymbol{\Sigma}_n \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \boldsymbol{\Sigma}_n \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}) \\
& + 4\boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \boldsymbol{\Sigma}_n \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{X}_n \boldsymbol{\beta}_0 \\
& + 4\boldsymbol{\beta}_0' \boldsymbol{X}_n' \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1} \mathrm{dg}(\boldsymbol{\Sigma}_n^{(3)} \odot \boldsymbol{R}_n^{-1\prime} \boldsymbol{G}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{G}_n \boldsymbol{R}_n^{-1}).
\end{aligned}$$

From Lemma 2 again, one sees that both $\mathrm{Var}(r_n)$ and $\mathrm{Var}(d_n)$ are $O(n)$. With these, it is obvious that $[r_n - \mathrm{E}(r_n)]/\mathrm{E}(r_n) = O_p(n^{-1/2})$ and $[d_n - \mathrm{E}(d_n)]/\mathrm{E}(d_n) = O_p(n^{-1/2})$. ■

Proof of Theorem 1

With r_n and d_n defined as in Lemma 5, note that

$$\begin{aligned}
& \sqrt{n}\left(\hat{\lambda}_n - \lambda_0 - \frac{\boldsymbol{y}_n' \boldsymbol{S}_n' \boldsymbol{R}_n' \boldsymbol{H}_n' \boldsymbol{D}_n \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{S}_n \boldsymbol{y}_n}{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n}\right) \\
& = \sqrt{n}\left(\hat{\lambda}_n - \lambda_0 - \frac{\tilde{\boldsymbol{v}}_n' \boldsymbol{D}_n \tilde{\boldsymbol{v}}_n}{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n}\right) \\
& = \sqrt{n}\left(\hat{\lambda}_n - \lambda_0 - \frac{\boldsymbol{v}_n \boldsymbol{D}_n \boldsymbol{v}_n}{\boldsymbol{y}_n' \boldsymbol{W}_n' \boldsymbol{R}_n' \boldsymbol{H}_n \boldsymbol{R}_n \boldsymbol{W}_n \boldsymbol{y}_n}\right) + o_p(1) \\
& = \sqrt{n}\left(\frac{r_n - \boldsymbol{v}_n \boldsymbol{D}_n \boldsymbol{v}_n}{d_n}\right) + o_p(1) \\
& = \sqrt{n}\left(\frac{r_n - \boldsymbol{v}_n' \boldsymbol{D}_n \boldsymbol{v}_n}{\mathrm{E}(d_n)}\right)\left(1 + \frac{d_n - \mathrm{E}(d_n)}{\mathrm{E}(d_n)}\right)^{-1} + o_p(1) \\
& = \sqrt{n}\left(\frac{r_n - \boldsymbol{v}_n' \boldsymbol{D}_n \boldsymbol{v}_n}{\mathrm{E}(d_n)}\right) + o_p(1) \\
& = \sqrt{n}\frac{\sqrt{\mathrm{Var}(r_n - \boldsymbol{v}_n' \boldsymbol{D}_n \boldsymbol{v}_n)}}{\mathrm{E}(d_n)} \frac{r_n - \boldsymbol{v}_n' \boldsymbol{D}_n \boldsymbol{v}_n}{\sqrt{\mathrm{Var}(r_n - \boldsymbol{v}_n' \boldsymbol{D}_n \boldsymbol{v}_n)}} + o_p(1),
\end{aligned}$$

where the second equality follows from $\tilde{\boldsymbol{v}}_n'\boldsymbol{D}_n\tilde{\boldsymbol{v}}_n - \boldsymbol{v}_n\boldsymbol{D}_n\boldsymbol{v}_n = O_p(1)$ (see Lemma 4) and the second last equality follows from $[d_n - \mathrm{E}(d_n)]/\mathrm{E}(d_n) = O_p(n^{-1/2})$ (see Lemma 5). Similarly,

$$
\begin{aligned}
&\sqrt{n}\left(\hat{\rho}_n - \rho_0 - \frac{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}{\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n}\right)\\
&= \sqrt{n}\left(\hat{\rho}_n - \rho_0 - \frac{\tilde{\boldsymbol{v}}_n'\boldsymbol{K}_n\tilde{\boldsymbol{v}}_n}{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n}\right)\\
&= \sqrt{n}\left(\frac{\tilde{\boldsymbol{v}}_n'\boldsymbol{F}\tilde{\boldsymbol{v}}_n}{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n} - \frac{\tilde{\boldsymbol{v}}_n'\boldsymbol{K}_n\tilde{\boldsymbol{v}}_n}{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n}\right)\\
&= \sqrt{n}\left(\frac{\tilde{\boldsymbol{v}}_n'\boldsymbol{L}_n\tilde{\boldsymbol{v}}_n}{\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n}\right)\\
&= \sqrt{n}\left(\frac{\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n}{\boldsymbol{v}_n'\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{v}_n}\right) + o_p(1)\\
&= \sqrt{n}\left(\frac{\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n}{\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)}\right)\left(1 + \frac{\boldsymbol{v}_n'\boldsymbol{F}'\boldsymbol{F}\boldsymbol{v}_n - \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)}{\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)}\right)^{-1} + o_p(1)\\
&= \sqrt{n}\frac{\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n}{\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)} + o_p(1)\\
&= \sqrt{n}\frac{\sqrt{\mathrm{Var}(\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n)}}{\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)}\frac{\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n}{\sqrt{\mathrm{Var}(\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n)}} + o_p(1),
\end{aligned}
$$

where the fourth last equality follows from $\tilde{\boldsymbol{v}}_n'\boldsymbol{L}_n\tilde{\boldsymbol{v}}_n = \boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n + O_p(1)$ and $\tilde{\boldsymbol{u}}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\tilde{\boldsymbol{u}}_n = \boldsymbol{u}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{u}_n + O_p(1) = \boldsymbol{v}_n'\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{v}_n + O_p(1)$ (see Lemma 4) and the second last equality follows from $[\boldsymbol{v}_n'\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{v}_n - \mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)]/\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n) = O_p(n^{-1/2})$ (see Lemma 4). Applying Lemma 1 to the quadratic forms (r_n in the expansion for $\hat{\lambda}_n - \lambda_0$ and $\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n$ in the expansion for $\hat{\rho}_n - \rho_0$) and their linear combinations yields immediately the asymptotic distribution (5), where $\xi_1 = \lim_{n\to\infty} n\mathrm{Var}(r_n - \boldsymbol{v}_n'\boldsymbol{D}_n\boldsymbol{v}_n)/[\mathrm{E}(d_n)]^2$, $\xi_2 = \lim_{n\to\infty} n\mathrm{Var}(\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n)/[\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)]^2$, and $\xi_{12} = \lim_{n\to\infty} n\mathrm{Cov}(r_n - \boldsymbol{v}_n'\boldsymbol{D}_n\boldsymbol{v}_n, \boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n)/\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)\mathrm{E}(d_n)$ with their expressions given in Assumption 6. ■

Proof of Theorem 2

One can apply the extended delta method of multivariate case as in Phillips (2012) to derive the asymptotic distribution result (6). For this purpose, the following condition is sufficient: for a given $\delta > 0$, if $s_n \to \infty$ and $s_n/\sqrt{n} \to 0$,

$$\sup_{||s_n((\lambda,\rho)'-(\lambda_0,\rho_0)'))||<\delta} \|\boldsymbol{B}_n(\boldsymbol{B}_n^{-1}(\lambda,\rho) - \boldsymbol{B}_n^{-1})\| \overset{a.s.}{\to} 0,$$

where $\boldsymbol{B}_n = ((b_{n,11}, b_{n,12})', (b_{n,21}, b_{n,22})')'$ is the Jacobian matrix associated with $\boldsymbol{b}_n(\lambda,\rho)$, evaluated at (λ_0, ρ_0). Since all matrix norms are equivalent, it is sufficient to consider $\|\boldsymbol{B}_n(\boldsymbol{B}_n^{-1}(\lambda,\rho) - \boldsymbol{B}_n^{-1})\|$ where the norm is sub-multiplicative (say, $\|\cdot\|_2$). Then

$$
\begin{aligned}
\|\boldsymbol{B}_n(\boldsymbol{B}_n^{-1}(\lambda,\rho) - \boldsymbol{B}_n^{-1})\| &\le \|\boldsymbol{B}_n\|\|(\boldsymbol{B}_n^{-1}(\lambda,\rho) - \boldsymbol{B}_n^{-1})\|\\
&= \|\boldsymbol{B}_n\|\|\boldsymbol{B}_n^{-1}(\lambda,\rho)(\boldsymbol{I}_2 - \boldsymbol{B}_n(\lambda,\rho)\boldsymbol{B}_n^{-1})\|\\
&\le \|\boldsymbol{B}_n\|\|\boldsymbol{B}_n^{-1}(\lambda,\rho)\|\|\boldsymbol{I}_2 - \boldsymbol{B}_n(\lambda,\rho)\boldsymbol{B}_n^{-1})\|\\
&= \|\boldsymbol{B}_n\|\|\boldsymbol{B}_n^{-1}(\lambda,\rho)\|\|(\boldsymbol{B}_n - \boldsymbol{B}_n(\lambda,\rho))\boldsymbol{B}_n^{-1}\|\\
&\le \|\boldsymbol{B}_n\|\|\boldsymbol{B}_n^{-1}(\lambda,\rho)\|\|\boldsymbol{B}_n - \boldsymbol{B}_n(\lambda,\rho)\|\|\boldsymbol{B}_n^{-1}\|.
\end{aligned}
$$

After some tedious algebra, the elements of the Jacobian matrix $\boldsymbol{B}_n$ are as follows:

$$b_{n,11} = (\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n)^{-1} \cdot (2\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n$$

$$
\begin{aligned}
&- \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_{n,\lambda}\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n) - 1,\\
b_{n,12} &= (\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n)^{-1}\cdot(\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_{n,\rho}\boldsymbol{R}_n\boldsymbol{y}_n\\
&\quad - \boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{M}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{M}_n\boldsymbol{y}_n + 2\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{M}_n'\boldsymbol{H}_n\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad - 2\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_{n,\rho}\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_{n,\rho}\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n)\\
&\quad - (\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n)^{-2}\cdot(\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_{n,\rho}\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n - 2\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{M}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n)\\
&\quad \cdot(\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{D}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n),\\
b_{n,21} &= (\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n)^{-1}\cdot(2\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad - \boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n)\\
&\quad + 2(\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n)^{-2}\cdot\boldsymbol{y}_n'\boldsymbol{W}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad \cdot(\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n),\\
b_{n,22} &= (\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n)^{-1}\cdot[\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_{n,\rho}\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad + \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_{n,\rho}\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{M}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{M}_n\boldsymbol{S}_n\boldsymbol{y}_n + \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad + \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n^2\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n + 2\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{M}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad - 2\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_{n,\rho}\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\mathrm{Dg}(\boldsymbol{F}_n^2)\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n]\\
&\quad - 2(\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n)^{-2}\cdot(\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_{n,\rho}\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n\\
&\quad - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{M}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n + \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{F}_n'\boldsymbol{F}_n^2\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n)\\
&\quad \cdot(\boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{R}_n^{-1\prime}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - \boldsymbol{y}_n'\boldsymbol{S}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n) - 1,
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{H}_{n,\rho} &= \boldsymbol{M}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n' + \boldsymbol{R}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{M}_n'\\
&\quad - \boldsymbol{R}_n\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'(\boldsymbol{M}_n'\boldsymbol{R}_n + \boldsymbol{R}_n'\boldsymbol{M}_n)\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n',\\
\boldsymbol{D}_{n,\lambda} &= \mathrm{Dg}(\boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n^2\boldsymbol{R}_n^{-1}),\\
\boldsymbol{D}_{n,\rho} &= \mathrm{Dg}(\boldsymbol{H}_{n,\rho}\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1} - \boldsymbol{H}_n\boldsymbol{M}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1} + \boldsymbol{H}_n\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{R}_n^{-1}\boldsymbol{F}_n).
\end{aligned}
$$

By substituting $\boldsymbol{y}_n = \boldsymbol{S}_n^{-1}\boldsymbol{X}_n\boldsymbol{\beta}_0 + \boldsymbol{S}_n^{-1}\boldsymbol{R}_n^{-1}\boldsymbol{v}_n$ and applying Lemmas 1–3, one can see that all the elements of $\boldsymbol{B}_n$ are bounded almost surely. (It also holds for $\boldsymbol{B}_n(\lambda,\rho)$ for $\lambda\in\Lambda$ and $\rho\in\mathrm{P}$ given Assumption 4.(ii).) This, together with Assumption 7.(ii), implies that $\|\boldsymbol{B}_n\|$, $\|\boldsymbol{B}_n^{-1}(\lambda,\rho)\|$, and $\|\boldsymbol{B}_n^{-1}\|$ are all bounded almost surely. So it is sufficient to show

$$
\sup_{||s_n((\lambda,\rho)'-(\lambda_0,\rho_0)'))||<\delta}\|(\boldsymbol{B}_n(\lambda,\rho)-\boldsymbol{B}_n)\| \overset{a.s.}{\to} 0.
$$

Note that

$$
\|(\boldsymbol{B}_n(\lambda,\rho)-\boldsymbol{B}_n)\| \le \left[\sup_{(\lambda^*,\rho^*)}\|\boldsymbol{B}_n'(\lambda^*,\rho^*)\|\right]\left\|\left(\begin{bmatrix}\lambda\\ \rho\end{bmatrix}-\begin{bmatrix}\lambda_0\\ \rho_0\end{bmatrix}\right)\right\|
$$

where $(\lambda^*,\rho^*)'$ lies between $(\lambda,\rho)'$ and $(\lambda_0,\rho_0)'$ and $\boldsymbol{B}_n'(\lambda,\rho)$ denotes the matrix derivative of $\boldsymbol{B}_n(\lambda,\rho)$ with respect to $(\lambda,\rho)'$. Applying again Lemmas 1–3, one can check that all the elements of $\boldsymbol{B}_n'(\lambda,\rho)$ are bounded almost surely for $\lambda\in\Lambda$ and $\rho\in\mathrm{P}$. It then follows that

$$
\begin{aligned}
&\sup_{||s_n((\lambda,\rho)'-(\lambda_0,\rho_0)'))||<\delta}\|(\boldsymbol{B}_n(\lambda,\rho)-\boldsymbol{B}_n)\|\\
&\le \sup_{||s_n((\lambda,\rho)'-(\lambda_0,\rho_0)'))||<\delta}\left[\sup_{(\lambda^*,\rho^*)}\|\boldsymbol{B}_n'(\lambda^*,\rho^*)\|\right]\left\|\left(\begin{bmatrix}\lambda\\ \rho\end{bmatrix}-\begin{bmatrix}\lambda_0\\ \rho_0\end{bmatrix}\right)\right\|\\
&\le \left|\frac{\delta}{s_n}\right|\left[\sup_{(\lambda^*,\rho^*)}\|\boldsymbol{B}_n'(\lambda^*,\rho^*)\|\right]
\end{aligned}
$$

$$\xrightarrow{a.s.} 0.$$

Then one can use this sufficient condition, together with (5), to derive the asymptotic distribution (6) by following Phillips (2012). ■

Proof of Theorem 3

By substituting $\boldsymbol{R}_n(\hat{\rho}_{II}) = \boldsymbol{R}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{M}_n$ and $\boldsymbol{S}_n(\hat{\lambda}_{II}) = \boldsymbol{S}_n - (\hat{\lambda}_{II} - \lambda_0)\boldsymbol{W}_n$ into (7), one has

$$\begin{aligned}\boldsymbol{X}_n'\boldsymbol{R}_n'(\hat{\rho}_{II})\boldsymbol{R}_n(\hat{\rho}_{II})\boldsymbol{X}_n &= \boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n + (\hat{\rho}_{II} - \rho_0)^2\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{X}_n \\ &\quad - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n\end{aligned}$$

and

$$\begin{aligned}&\boldsymbol{X}_n'\boldsymbol{R}_n'(\hat{\rho}_{II})\boldsymbol{R}_n(\hat{\rho}_{II})\boldsymbol{S}_n(\hat{\lambda}_{II})\boldsymbol{y}_n \\ &= [\boldsymbol{X}_n'\boldsymbol{R}_n' - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{M}_n'][\boldsymbol{R}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{M}_n][\boldsymbol{S}_n - (\hat{\lambda}_{II} - \lambda_0)\boldsymbol{W}_n]\boldsymbol{y}_n \\ &= \boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - (\hat{\lambda}_{II} - \lambda_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &+ (\hat{\lambda}_{II} - \lambda_0)(\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n\boldsymbol{W}_n\boldsymbol{y}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &+ (\hat{\lambda}_{II} - \lambda_0)(\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n + (\hat{\rho}_{II} - \rho_0)^2\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &- (\hat{\lambda}_{II} - \lambda_0)(\hat{\rho}_{II} - \rho_0)^2\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{M}_n\boldsymbol{W}_n\boldsymbol{y}_n.\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{II} &= [\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n]^{-1} \\ &\quad \cdot [\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - (\hat{\lambda}_{II} - \lambda_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &\qquad - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n] + o_p(n^{-1/2}) \\ &= [(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1} \\ &\qquad + (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1} + o_p(n^{-3/2})] \\ &\quad \cdot [\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n - (\hat{\lambda}_{II} - \lambda_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &\qquad - (\hat{\rho}_{II} - \rho_0)\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n] + o_p(n^{-1/2}) \\ &= (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &\quad + (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &\quad - (\hat{\lambda}_{II} - \lambda_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{W}_n\boldsymbol{y}_n \\ &\quad - (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n\boldsymbol{S}_n\boldsymbol{y}_n \\ &\quad - (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{R}_n\boldsymbol{S}_n\boldsymbol{y}_n + o_p(n^{-1/2}) \\ &= \boldsymbol{\beta}_0 + (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{v}_n \\ &\quad + (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n\boldsymbol{\beta}_0 \\ &\quad + (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{v}_n \\ &\quad - (\hat{\lambda}_{II} - \lambda_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n(\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0 + \boldsymbol{G}_n\boldsymbol{R}_n^{-1}\boldsymbol{v}_n) \\ &\quad - (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n(\boldsymbol{X}_n\boldsymbol{\beta}_0 + \boldsymbol{R}_n^{-1}\boldsymbol{v}_n) \\ &\quad - (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{M}_n'(\boldsymbol{R}_n\boldsymbol{X}_n\boldsymbol{\beta}_0 + \boldsymbol{v}_n) + o_p(n^{-1/2}) \\ &= \boldsymbol{\beta}_0 + (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{v}_n \\ &\quad + (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'(\boldsymbol{R}_n'\boldsymbol{M}_n + \boldsymbol{M}_n'\boldsymbol{R}_n)\boldsymbol{X}_n\boldsymbol{\beta}_0 \\ &\quad - (\hat{\lambda}_{II} - \lambda_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0\end{aligned}$$

$$
\begin{aligned}
&- (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{M}_n\boldsymbol{X}_n\boldsymbol{\beta}_0 \\
&- (\hat{\rho}_{II} - \rho_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{M}_n'\boldsymbol{R}_n\boldsymbol{X}_n\boldsymbol{\beta}_0 + o_p(n^{-1/2}) \\
= &\, \boldsymbol{\beta}_0 + (\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{v}_n \\
&- (\hat{\lambda}_{II} - \lambda_0)(\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{X}_n)^{-1}\boldsymbol{X}_n'\boldsymbol{R}_n'\boldsymbol{R}_n\boldsymbol{G}_n\boldsymbol{X}_n\boldsymbol{\beta}_0 + o_p(n^{-1/2}). \qquad (1)
\end{aligned}
$$

From the definition of $(\hat{\lambda}_{II}, \hat{\rho}_{II})'$,

$$
\sqrt{n}\begin{pmatrix} \hat{\lambda}_{II} - \lambda_0 \\ \hat{\rho}_{II} - \rho_0 \end{pmatrix} = \boldsymbol{B}_n^{-1}\sqrt{n}\begin{pmatrix} \frac{\boldsymbol{v}_n'\boldsymbol{E}_n\boldsymbol{v}_n + \boldsymbol{\beta}_0'\boldsymbol{X}_n'\boldsymbol{G}_n'\boldsymbol{R}_n'\boldsymbol{H}_n\boldsymbol{v}_n}{\mathrm{E}(d_n)} \\ \frac{\boldsymbol{v}_n'\boldsymbol{L}_n\boldsymbol{v}_n}{\mathrm{tr}(\boldsymbol{\Sigma}_n\boldsymbol{F}_n'\boldsymbol{F}_n)} \end{pmatrix} + o_p(1). \qquad (2)
$$

In view of (1) and (2), one can write any linear combination of $(\hat{\lambda}_{II} - \lambda_0, \hat{\rho}_{II} - \rho_0, (\hat{\boldsymbol{\beta}}_{II} - \boldsymbol{\beta}_0)')'$ as a form satisfying Q_n in Lemma 1. It implies that $\sqrt{n}(\hat{\lambda}_{II} - \lambda_0, \hat{\rho}_{II} - \rho_0, (\hat{\boldsymbol{\beta}}_{II} - \boldsymbol{\beta}_0)')'$ has a joint asymptotic normal distribution, given by (8). ■

Appendix B: Additional Simulation Results

This appendix collects additional simulation results, all based on 10,000 simulations. Table 1.A1 supplements Table 1 when λ_0 is negative. Tables 2.A1-A3 supplements Table 2 with other parameter configurations and Tables 2.A4-A7 report results under a larger sample size $1,000$. Tables 2.A8-A11 present results when $n = 1,000$ and J is fixed (10, 20, and 100). In Table 2 and Tables 2.A1-A7, when n increases, the relative density of the spatial weight matrices stay the same, whereas in Tables 2.A8-A.11, as n goes up, the weight matrices become less dense. One can see that when the spatial weight matrices are dense, increasing the sample size does not help much in improving the performance of the three estimators unless the sample size increases much faster than J. For example, comparing Table 2 and Tables 2.A.4, one sees that when n goes up from 200 to 1,000 (but the proportionality of J relative to n stays the same), all the three estimators make little improvement, though it is still the case the II estimator performs relatively better. On the other hand, by comparing Table 2 and Table 2.A.8, one sees that as the sample size increases, given fixed J, the spatial weight matrices become less dense and the three estimators in general perform better in terms of reduced bias and lower RMSE. One also observes that when $n = 1,000$, $J = 100$ (so that $\boldsymbol{W}_n$ and $\boldsymbol{M}_n$ have 10% non-zero entries), and $\rho_0 \geq 0$, there still exist non-negligible size distortions for testing λ_0, ρ_0, and β_{10} across the three procedures. Table A.12 reports additional results when $n = 5,000$, $J = 100$, and $\rho_0 \geq 0$. Now as the weight matrices become less dense, the three procedures have improved size performances.

Table 1.A1: GMM, GS2SLS, and II under County Contiguity Weight Matrices ($n = 761$, $\lambda_0 < 0$)

	GMM			GS2SLS			II		
$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$\lambda_0 = -0.9$	0.006	0.077	4.7%	0.017	0.106	5.9%	0.013	0.063	2.8%
$\rho_0 = 0.6$	-0.009	0.062	7.1%	-0.019	0.077	11.9%	-0.014	0.055	4.3%
$\beta_{10} = 0.8$	-0.006	0.256	6.5%	-0.019	0.299	5.9%	-0.017	0.243	5.1%
$\beta_{20} = 0.2$	0.000	0.056	5.2%	0.000	0.066	4.9%	0.000	0.056	4.9%
$\beta_{30} = 1.5$	0.001	0.064	7.0%	0.001	0.074	5.2%	0.005	0.058	4.8%
$\lambda_0 = -0.9$	0.007	0.078	4.4%	0.006	0.086	5.5%	0.015	0.066	2.8%
$\rho_0 = 0.3$	-0.013	0.084	6.2%	-0.014	0.088	7.0%	-0.019	0.078	4.4%
$\beta_{10} = 0.8$	-0.010	0.229	6.5%	-0.006	0.234	5.3%	-0.022	0.215	4.9%
$\beta_{20} = 0.2$	0.001	0.059	6.1%	0.000	0.060	5.4%	0.001	0.058	5.3%
$\beta_{30} = 1.5$	0.001	0.058	5.8%	0.000	0.060	5.0%	0.005	0.055	4.7%
$\lambda_0 = -0.9$	0.007	0.077	3.7%	0.002	0.080	4.9%	0.017	0.068	2.8%
$\rho_0 = 0.0$	-0.013	0.096	5.1%	-0.009	0.096	4.9%	-0.022	0.092	4.2%
$\beta_{10} = 0.8$	-0.010	0.225	7.5%	-0.003	0.213	5.2%	-0.023	0.205	4.8%
$\beta_{20} = 0.2$	0.001	0.061	6.8%	0.000	0.058	5.2%	0.001	0.058	5.3%
$\beta_{30} = 1.5$	0.000	0.054	5.3%	-0.001	0.055	5.2%	0.003	0.053	5.0%
$\lambda_0 = -0.9$	0.005	0.077	3.0%	-0.002	0.081	5.4%	0.018	0.071	2.9%
$\rho_0 = -0.3$	-0.012	0.106	5.8%	-0.005	0.107	4.7%	-0.025	0.102	4.7%
$\beta_{10} = 0.8$	-0.005	0.222	8.5%	0.006	0.207	5.3%	-0.020	0.196	4.7%
$\beta_{20} = 0.2$	0.000	0.063	7.9%	-0.001	0.058	5.1%	-0.001	0.057	5.2%
$\beta_{30} = 1.5$	-0.001	0.052	5.3%	-0.002	0.052	5.1%	0.000	0.051	4.9%
$\lambda_0 = -0.9$	0.005	0.078	2.8%	-0.004	0.088	5.6%	0.021	0.074	3.0%
$\rho_0 = -0.6$	-0.010	0.106	5.0%	0.002	0.112	4.1%	-0.025	0.103	5.0%
$\beta_{10} = 0.8$	-0.005	0.225	9.8%	0.009	0.214	5.2%	-0.021	0.190	4.7%
$\beta_{20} = 0.2$	0.000	0.065	9.6%	-0.001	0.062	5.0%	0.000	0.056	5.1%
$\beta_{30} = 1.5$	-0.002	0.051	5.3%	-0.004	0.051	5.2%	-0.004	0.050	5.2%
$\lambda_0 = -0.4$	0.001	0.094	10.8%	0.178	0.295	23.7%	0.011	0.076	4.7%
$\rho_0 = 0.9$	-0.004	0.032	11.1%	-0.072	0.115	42.9%	-0.008	0.025	4.2%
$\beta_{10} = 0.8$	0.004	1.308	6.8%	-0.269	0.765	16.4%	-0.005	0.607	6.5%
$\beta_{20} = 0.2$	0.000	0.055	5.7%	0.002	0.094	6.8%	0.000	0.055	5.4%
$\beta_{30} = 1.5$	0.000	0.064	8.0%	0.008	0.100	7.6%	0.004	0.057	5.0%
$\lambda_0 = -0.4$	0.004	0.103	10.1%	0.023	0.126	7.5%	0.011	0.088	5.3%
$\rho_0 = 0.6$	-0.010	0.074	9.3%	-0.024	0.089	10.8%	-0.014	0.066	5.3%
$\beta_{10} = 0.8$	-0.008	0.295	7.5%	-0.038	0.336	6.0%	-0.021	0.275	5.1%
$\beta_{20} = 0.2$	0.001	0.056	5.5%	0.001	0.066	5.1%	0.001	0.056	5.4%
$\beta_{30} = 1.5$	0.000	0.061	7.0%	-0.001	0.066	5.1%	0.003	0.058	5.1%
$\lambda_0 = -0.4$	0.000	0.090	7.3%	0.004	0.086	5.3%	0.007	0.085	5.8%
$\rho_0 = 0.0$	-0.009	0.104	6.7%	-0.013	0.101	4.8%	-0.015	0.102	6.1%
$\beta_{10} = 0.8$	0.003	0.250	7.8%	-0.003	0.231	5.3%	-0.008	0.231	5.5%
$\beta_{20} = 0.2$	-0.001	0.061	7.1%	-0.001	0.058	5.1%	-0.001	0.058	5.5%
$\beta_{30} = 1.5$	-0.001	0.052	5.1%	0.000	0.052	4.9%	0.000	0.052	4.9%
$\lambda_0 = -0.4$	-0.001	0.085	6.7%	-0.003	0.092	5.6%	0.005	0.083	6.0%
$\rho_0 = -0.6$	-0.005	0.109	7.0%	-0.001	0.114	5.2%	-0.009	0.105	6.1%
$\beta_{10} = 0.8$	0.005	0.244	10.8%	0.009	0.228	5.4%	-0.005	0.205	5.4%
$\beta_{20} = 0.2$	-0.001	0.065	9.7%	-0.001	0.062	5.3%	-0.001	0.056	5.5%
$\beta_{30} = 1.5$	-0.002	0.052	5.2%	-0.002	0.053	5.2%	-0.003	0.052	5.1%
$\lambda_0 = -0.4$	-0.010	0.077	4.0%	-0.012	0.103	6.1%	-0.014	0.073	3.6%
$\rho_0 = -0.9$	0.006	0.090	2.8%	0.022	0.104	3.2%	0.020	0.080	2.1%
$\beta_{10} = 0.8$	0.020	0.241	11.6%	0.024	0.248	5.1%	0.019	0.189	4.4%
$\beta_{20} = 0.2$	-0.002	0.067	12.3%	-0.001	0.068	4.9%	0.000	0.054	5.0%
$\beta_{30} = 1.5$	0.000	0.054	5.0%	-0.004	0.056	4.9%	0.003	0.054	4.8%
$\lambda_0 = -0.1$	0.003	0.103	10.7%	0.190	0.301	27.6%	0.016	0.083	4.8%
$\rho_0 = 0.9$	-0.005	0.037	11.6%	-0.089	0.139	37.8%	-0.010	0.029	3.9%
$\beta_{10} = 0.8$	0.003	2.805	7.1%	-0.367	0.838	21.9%	-0.032	0.622	6.6%
$\beta_{20} = 0.2$	-0.001	0.054	5.4%	-0.001	0.089	6.6%	0.000	0.054	5.3%
$\beta_{30} = 1.5$	0.000	0.063	7.9%	-0.002	0.086	5.7%	0.005	0.056	5.1%
$\lambda_0 = -0.1$	0.002	0.107	9.8%	0.023	0.129	7.8%	0.014	0.096	6.0%
$\rho_0 = 0.6$	-0.010	0.080	9.3%	-0.026	0.095	10.4%	-0.018	0.074	5.5%
$\beta_{10} = 0.8$	-0.001	0.326	7.3%	-0.040	0.368	6.1%	-0.026	0.302	5.0%
$\beta_{20} = 0.2$	0.000	0.056	5.3%	0.000	0.065	5.1%	0.000	0.056	4.9%
$\beta_{30} = 1.5$	0.000	0.059	6.8%	-0.002	0.062	4.9%	0.003	0.057	5.4%
$\lambda_0 = -0.1$	-0.002	0.087	7.0%	0.003	0.084	5.6%	0.005	0.084	6.4%
$\rho_0 = 0.0$	-0.008	0.105	6.4%	-0.013	0.103	4.6%	-0.013	0.104	5.8%
$\beta_{10} = 0.8$	0.006	0.268	7.7%	-0.003	0.247	5.2%	-0.007	0.247	5.5%
$\beta_{20} = 0.2$	0.000	0.060	6.6%	0.000	0.057	5.1%	0.000	0.057	5.2%
$\beta_{30} = 1.5$	-0.002	0.052	5.3%	-0.001	0.052	5.2%	-0.001	0.052	5.2%
$\lambda_0 = -0.1$	-0.003	0.077	6.3%	-0.003	0.084	5.6%	0.003	0.075	5.6%
$\rho_0 = -0.6$	-0.006	0.107	7.2%	-0.004	0.114	5.6%	-0.010	0.101	6.1%
$\beta_{10} = 0.8$	0.008	0.254	10.4%	0.009	0.239	5.4%	-0.003	0.213	5.2%
$\beta_{20} = 0.2$	0.000	0.065	9.9%	0.000	0.062	5.1%	0.000	0.056	5.2%
$\beta_{30} = 1.5$	-0.003	0.054	5.3%	-0.003	0.055	5.0%	-0.004	0.054	5.1%
$\lambda_0 = -0.1$	-0.009	0.070	4.4%	-0.010	0.094	5.7%	-0.011	0.066	3.7%
$\rho_0 = -0.9$	0.005	0.086	2.7%	0.019	0.101	3.6%	0.016	0.075	2.0%
$\beta_{10} = 0.8$	0.019	0.251	11.5%	0.021	0.262	5.6%	0.017	0.198	4.6%
$\beta_{20} = 0.2$	0.000	0.068	12.4%	0.000	0.069	5.4%	0.001	0.054	5.2%
$\beta_{30} = 1.5$	0.001	0.056	5.0%	-0.002	0.060	4.9%	0.003	0.055	4.6%

Table 2.A1: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 200$, $\lambda_0 = -0.4$, $\rho_0 \geq 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 10$	$\lambda_0 = -0.4$	-0.010	0.275	8.2%	0.865	0.994	61.9%	0.147	0.352	5.8%
	$\rho_0 = 0.9$	-0.007	0.079	18.3%	-0.468	0.639	39.5%	-0.070	0.141	2.9%
	$\beta_{10} = 0.8$	1.448	116.901	6.7%	-0.894	1.246	44.1%	-0.164	1.151	10.1%
	$\beta_{20} = 0.2$	-0.001	0.108	5.9%	0.008	0.129	4.6%	0.003	0.110	5.1%
	$\beta_{30} = 1.5$	-0.009	0.110	7.2%	0.060	0.139	9.2%	0.025	0.114	5.1%
	$\lambda_0 = -0.4$	-0.016	0.298	11.9%	0.307	0.505	28.2%	0.124	0.327	10.3%
	$\rho_0 = 0.7$	-0.013	0.163	16.8%	-0.220	0.330	13.0%	-0.113	0.217	6.2%
	$\beta_{10} = 0.8$	-1.085	112.599	6.7%	-0.315	0.717	15.2%	-0.135	0.599	7.1%
	$\beta_{20} = 0.2$	-0.001	0.112	7.1%	0.002	0.124	5.5%	0.003	0.113	5.8%
	$\beta_{30} = 1.5$	-0.009	0.110	7.0%	0.012	0.118	5.1%	0.017	0.107	4.9%
	$\lambda_0 = -0.4$	-0.023	0.285	13.5%	0.066	0.294	11.3%	0.058	0.266	11.5%
	$\rho_0 = 0.3$	-0.016	0.262	16.3%	-0.127	0.266	7.0%	-0.119	0.273	9.7%
	$\beta_{10} = 0.8$	1.950	66.564	7.6%	-0.055	0.486	6.6%	-0.048	0.467	6.9%
	$\beta_{20} = 0.2$	-0.004	0.115	7.4%	-0.003	0.116	6.1%	-0.002	0.115	6.4%
	$\beta_{30} = 1.5$	-0.011	0.106	7.1%	-0.004	0.104	5.4%	0.000	0.101	5.3%
	$\lambda_0 = -0.4$	-0.026	0.275	13.6%	0.031	0.263	9.0%	0.036	0.249	11.5%
	$\rho_0 = 0.1$	-0.017	0.288	15.4%	-0.092	0.270	7.0%	-0.113	0.286	10.5%
	$\beta_{10} = 0.8$	-0.980	56.210	8.3%	-0.031	0.460	6.2%	-0.035	0.452	7.2%
	$\beta_{20} = 0.2$	-0.001	0.117	7.4%	-0.001	0.115	5.9%	-0.001	0.115	6.4%
	$\beta_{30} = 1.5$	-0.010	0.104	6.8%	-0.004	0.101	5.4%	-0.003	0.100	5.2%
	$\lambda_0 = -0.4$	-0.023	0.266	13.4%	0.030	0.248	8.1%	0.033	0.232	10.5%
	$\rho_0 = 0.0$	-0.015	0.299	14.6%	-0.081	0.271	6.7%	-0.106	0.285	9.5%
	$\beta_{10} = 0.8$	0.244	36.282	8.0%	-0.026	0.440	5.8%	-0.029	0.430	6.2%
	$\beta_{20} = 0.2$	-0.002	0.114	6.4%	-0.002	0.112	5.2%	-0.002	0.113	5.8%
	$\beta_{30} = 1.5$	-0.012	0.103	7.1%	-0.006	0.100	5.5%	-0.006	0.100	5.4%
$J = 20$	$\lambda_0 = -0.4$	0.010	0.403	5.7%	1.068	1.172	74.7%	0.312	0.523	13.1%
	$\rho_0 = 0.9$	-0.028	0.165	29.1%	-0.711	0.850	46.6%	-0.170	0.272	5.0%
	$\beta_{10} = 0.8$	-3.154	282.471	8.8%	-1.090	1.337	60.1%	-0.319	1.119	15.8%
	$\beta_{20} = 0.2$	0.002	0.111	6.4%	0.008	0.119	5.3%	0.006	0.113	5.2%
	$\beta_{30} = 1.5$	-0.006	0.106	7.0%	0.043	0.117	7.4%	0.025	0.107	4.9%
	$\lambda_0 = -0.4$	-0.014	0.422	8.8%	0.554	0.773	42.5%	0.233	0.457	17.2%
	$\rho_0 = 0.7$	-0.035	0.280	26.0%	-0.440	0.593	21.8%	-0.240	0.379	10.3%
	$\beta_{10} = 0.8$	-2.309	291.061	7.9%	-0.561	0.901	27.9%	-0.235	0.669	11.3%
	$\beta_{20} = 0.2$	-0.002	0.111	6.5%	0.002	0.116	5.1%	0.001	0.112	5.3%
	$\beta_{30} = 1.5$	-0.009	0.104	6.6%	0.015	0.107	4.9%	0.012	0.101	4.9%
	$\lambda_0 = -0.4$	-0.041	0.394	9.2%	0.159	0.446	15.9%	0.105	0.356	15.0%
	$\rho_0 = 0.3$	-0.014	0.390	22.2%	-0.217	0.437	11.1%	-0.238	0.422	12.3%
	$\beta_{10} = 0.8$	2.515	232.256	7.0%	-0.155	0.590	8.9%	-0.099	0.522	8.3%
	$\beta_{20} = 0.2$	-0.003	0.115	6.8%	-0.002	0.114	5.5%	-0.002	0.114	5.7%
	$\beta_{30} = 1.5$	-0.010	0.103	6.6%	-0.001	0.101	5.1%	0.000	0.100	5.4%
	$\lambda_0 = -0.4$	-0.041	0.376	9.1%	0.084	0.388	10.5%	0.066	0.326	13.0%
	$\rho_0 = 0.1$	-0.009	0.425	20.2%	-0.144	0.413	8.6%	-0.223	0.423	11.5%
	$\beta_{10} = 0.8$	3.432	193.001	7.1%	-0.082	0.542	6.5%	-0.063	0.498	7.5%
	$\beta_{20} = 0.2$	-0.001	0.115	6.9%	0.000	0.114	5.7%	-0.001	0.115	6.2%
	$\beta_{30} = 1.5$	-0.007	0.102	6.5%	-0.001	0.100	5.1%	-0.001	0.099	5.2%
	$\lambda_0 = -0.4$	-0.043	0.364	8.8%	0.047	0.362	7.8%	0.046	0.307	11.3%
	$\rho_0 = 0.0$	-0.006	0.439	19.4%	-0.103	0.393	7.1%	-0.205	0.417	11.3%
	$\beta_{10} = 0.8$	1.639	178.446	6.9%	-0.044	0.514	5.3%	-0.043	0.475	6.7%
	$\beta_{20} = 0.2$	-0.002	0.115	7.0%	-0.001	0.113	5.4%	-0.001	0.114	6.1%
	$\beta_{30} = 1.5$	-0.008	0.101	6.6%	-0.002	0.100	5.6%	-0.004	0.099	5.7%
$J = 100$	$\lambda_0 = -0.4$	0.374	0.826	21.2%	1.066	1.504	41.2%	0.662	0.891	35.5%
	$\rho_0 = 0.9$	-0.165	0.577	63.8%	-0.806	1.053	27.4%	-1.248	1.377	30.6%
	$\beta_{10} = 0.8$	11.659	579.236	13.3%	-1.070	1.801	39.2%	-0.672	1.222	38.0%
	$\beta_{20} = 0.2$	-0.001	0.114	6.6%	0.001	0.114	5.5%	0.000	0.114	5.7%
	$\beta_{30} = 1.5$	0.006	0.101	6.8%	0.014	0.100	5.2%	0.008	0.100	5.6%
	$\lambda_0 = -0.4$	0.262	0.792	20.1%	0.839	1.376	30.5%	0.469	0.763	25.1%
	$\rho_0 = 0.7$	-0.081	0.649	57.8%	-0.611	0.924	25.4%	-1.146	1.275	26.1%
	$\beta_{10} = 0.8$	8.415	561.663	12.3%	-0.848	1.467	26.4%	-0.472	0.895	20.7%
	$\beta_{20} = 0.2$	0.001	0.114	6.4%	0.001	0.114	5.7%	0.001	0.114	5.5%
	$\beta_{30} = 1.5$	0.006	0.099	5.9%	0.015	0.101	5.7%	0.009	0.099	5.3%
	$\lambda_0 = -0.4$	0.115	0.737	17.4%	0.440	1.225	14.3%	0.204	0.593	12.3%
	$\rho_0 = 0.3$	0.083	0.777	48.7%	-0.217	0.727	24.1%	-0.912	1.032	18.1%
	$\beta_{10} = 0.8$	-2.982	518.406	12.2%	-0.441	1.284	11.5%	-0.204	0.703	9.1%
	$\beta_{20} = 0.2$	-0.001	0.115	6.5%	-0.001	0.115	5.8%	-0.001	0.114	5.4%
	$\beta_{30} = 1.5$	0.002	0.099	6.3%	0.008	0.099	5.2%	0.003	0.098	5.1%
	$\lambda_0 = -0.4$	0.100	0.723	16.8%	0.220	1.187	8.6%	0.103	0.544	9.3%
	$\rho_0 = 0.1$	0.134	0.828	43.0%	0.047	0.671	27.7%	-0.782	0.899	12.6%
	$\beta_{10} = 0.8$	7.076	483.110	11.8%	-0.224	1.268	7.6%	-0.104	0.658	6.7%
	$\beta_{20} = 0.2$	0.000	0.115	6.8%	0.001	0.114	5.7%	0.000	0.115	5.9%
	$\beta_{30} = 1.5$	0.000	0.098	6.1%	0.005	0.099	5.0%	0.001	0.097	5.1%
	$\lambda_0 = -0.4$	0.077	0.714	16.2%	0.081	1.172	6.9%	0.061	0.524	8.3%
	$\rho_0 = 0.0$	0.187	0.839	41.2%	0.168	0.685	29.6%	-0.715	0.829	10.0%
	$\beta_{10} = 0.8$	-1.303	468.419	11.4%	-0.085	1.252	6.1%	-0.060	0.635	5.6%
	$\beta_{20} = 0.2$	-0.001	0.115	6.8%	0.001	0.115	6.0%	0.000	0.114	5.6%
	$\beta_{30} = 1.5$	-0.001	0.099	6.1%	0.002	0.099	4.9%	-0.001	0.098	5.2%

Table 2.A2: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 200$, $\lambda_0 = 0.4$, $\rho_0 < 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J=10$	$\lambda_0=0.4$	-0.025	0.106	4.0%	-0.020	0.121	4.7%	-0.209	0.434	3.3%
	$\rho_0=-0.9$	0.052	0.200	2.6%	0.128	0.240	4.2%	0.278	0.602	2.0%
	$\beta_{10}=0.8$	0.057	0.412	5.7%	0.048	0.452	5.0%	0.487	1.090	4.1%
	$\beta_{20}=0.2$	0.000	0.112	6.3%	0.000	0.122	5.5%	0.000	0.112	4.7%
	$\beta_{30}=1.5$	0.002	0.106	6.2%	0.003	0.109	5.9%	-0.005	0.122	5.1%
	$\lambda_0=0.4$	-0.013	0.118	6.0%	-0.008	0.120	4.7%	-0.109	0.312	3.4%
	$\rho_0=-0.7$	-0.003	0.249	3.1%	0.055	0.241	3.6%	0.107	0.437	1.9%
	$\beta_{10}=0.8$	0.117	8.682	6.7%	0.016	0.452	5.5%	0.256	0.819	4.8%
	$\beta_{20}=0.2$	0.000	0.115	7.2%	0.000	0.121	6.2%	0.000	0.114	5.9%
	$\beta_{30}=1.5$	-0.001	0.103	6.1%	0.000	0.104	5.3%	-0.006	0.109	4.4%
	$\lambda_0=0.4$	-0.024	0.168	9.0%	0.006	0.135	6.5%	-0.014	0.160	8.0%
	$\rho_0=-0.3$	-0.016	0.298	9.8%	-0.015	0.268	5.9%	-0.065	0.284	8.2%
	$\beta_{10}=0.8$	0.077	33.499	7.6%	-0.006	0.474	5.4%	0.043	0.516	6.4%
	$\beta_{20}=0.2$	-0.003	0.117	7.8%	-0.002	0.116	6.2%	-0.003	0.115	6.9%
	$\beta_{30}=1.5$	-0.005	0.102	6.6%	-0.005	0.100	5.6%	-0.006	0.101	5.4%
	$\lambda_0=0.4$	-0.029	0.180	10.0%	0.005	0.143	7.1%	-0.011	0.153	8.7%
	$\rho_0=-0.2$	-0.016	0.300	10.4%	-0.017	0.279	6.9%	-0.076	0.279	8.8%
	$\beta_{10}=0.8$	0.185	30.476	8.0%	-0.010	0.489	5.3%	0.029	0.509	6.3%
	$\beta_{20}=0.2$	-0.001	0.117	7.6%	-0.001	0.115	5.8%	-0.001	0.115	6.5%
	$\beta_{30}=1.5$	-0.005	0.102	6.7%	-0.004	0.100	5.4%	-0.005	0.100	5.5%
	$\lambda_0=0.4$	-0.032	0.197	11.1%	0.012	0.147	8.0%	-0.004	0.153	9.8%
	$\rho_0=-0.1$	-0.013	0.300	10.6%	-0.030	0.275	6.8%	-0.084	0.276	8.8%
	$\beta_{10}=0.8$	0.186	38.113	7.8%	-0.025	0.494	5.6%	0.012	0.504	6.6%
	$\beta_{20}=0.2$	-0.002	0.115	6.7%	-0.002	0.113	5.4%	-0.001	0.113	6.0%
	$\beta_{30}=1.5$	-0.009	0.102	6.9%	-0.007	0.099	5.7%	-0.008	0.100	5.6%
$J=20$	$\lambda_0=0.4$	-0.051	0.201	5.1%	-0.011	0.168	4.1%	-0.230	0.444	3.1%
	$\rho_0=-0.9$	0.128	0.349	5.0%	0.164	0.344	4.3%	0.265	0.588	2.7%
	$\beta_{10}=0.8$	-0.363	83.794	5.5%	0.021	0.522	4.5%	0.533	1.105	3.6%
	$\beta_{20}=0.2$	0.001	0.115	6.5%	0.002	0.117	5.5%	0.002	0.113	4.7%
	$\beta_{30}=1.5$	0.000	0.105	6.8%	-0.001	0.105	5.8%	0.004	0.110	5.3%
	$\lambda_0=0.4$	-0.050	0.231	7.7%	-0.006	0.180	5.4%	-0.143	0.347	3.7%
	$\rho_0=-0.7$	0.061	0.388	5.4%	0.087	0.362	4.4%	0.083	0.458	2.2%
	$\beta_{10}=0.8$	-0.312	104.780	6.4%	0.014	0.545	4.9%	0.333	0.896	4.3%
	$\beta_{20}=0.2$	-0.001	0.115	6.9%	0.000	0.115	5.6%	0.000	0.113	5.7%
	$\beta_{30}=1.5$	-0.004	0.101	6.4%	-0.004	0.101	5.3%	-0.001	0.103	5.0%
	$\lambda_0=0.4$	-0.062	0.293	11.9%	0.019	0.197	7.3%	-0.046	0.243	8.0%
	$\rho_0=-0.3$	0.010	0.450	8.9%	-0.021	0.405	4.4%	-0.113	0.399	5.1%
	$\beta_{10}=0.8$	0.225	144.841	8.0%	-0.037	0.578	4.9%	0.117	0.678	5.9%
	$\beta_{20}=0.2$	-0.003	0.116	6.8%	-0.002	0.115	5.7%	-0.003	0.114	5.9%
	$\beta_{30}=1.5$	-0.007	0.101	6.4%	-0.004	0.099	5.4%	-0.004	0.100	5.5%
	$\lambda_0=0.4$	-0.068	0.316	14.7%	0.023	0.211	8.6%	-0.034	0.234	9.4%
	$\rho_0=-0.2$	0.004	0.456	13.8%	-0.039	0.418	7.2%	-0.140	0.400	8.2%
	$\beta_{10}=0.8$	-2.658	155.756	8.9%	-0.050	0.610	5.7%	0.084	0.662	6.8%
	$\beta_{20}=0.2$	-0.002	0.115	6.9%	-0.001	0.114	5.5%	-0.001	0.114	6.0%
	$\beta_{30}=1.5$	-0.006	0.102	6.7%	-0.003	0.099	5.4%	-0.004	0.100	5.4%
	$\lambda_0=0.4$	-0.075	0.325	14.8%	0.031	0.218	9.5%	-0.028	0.234	10.7%
	$\rho_0=-0.1$	0.004	0.455	16.1%	-0.056	0.425	9.1%	-0.149	0.398	9.8%
	$\beta_{10}=0.8$	-0.297	183.371	8.8%	-0.069	0.622	6.0%	0.068	0.660	7.0%
	$\beta_{20}=0.2$	-0.002	0.116	7.0%	-0.001	0.114	5.6%	-0.001	0.114	6.1%
	$\beta_{30}=1.5$	-0.009	0.102	6.9%	-0.005	0.100	5.7%	-0.006	0.100	5.9%
$J=100$	$\lambda_0=0.4$	-0.188	0.671	18.7%	0.039	0.810	5.0%	-0.324	0.648	13.9%
	$\rho_0=-0.9$	0.554	0.992	24.8%	0.448	0.799	13.3%	0.081	0.396	1.2%
	$\beta_{10}=0.8$	2.623	353.489	15.4%	-0.085	1.926	4.5%	0.756	1.559	11.3%
	$\beta_{20}=0.2$	-0.002	0.115	6.6%	-0.001	0.115	5.8%	-0.001	0.114	5.3%
	$\beta_{30}=1.5$	-0.003	0.100	6.6%	0.000	0.099	5.7%	-0.001	0.099	5.5%
	$\lambda_0=0.4$	-0.153	0.666	20.9%	0.092	0.802	6.5%	-0.281	0.620	13.2%
	$\rho_0=-0.7$	0.441	0.951	26.8%	0.284	0.736	13.3%	-0.098	0.414	0.9%
	$\beta_{10}=0.8$	0.611	373.715	17.5%	-0.219	1.908	5.7%	0.655	1.503	10.8%
	$\beta_{20}=0.2$	0.000	0.115	6.7%	0.001	0.114	5.7%	0.001	0.114	5.7%
	$\beta_{30}=1.5$	-0.001	0.099	6.0%	0.003	0.099	5.2%	0.001	0.098	4.8%
	$\lambda_0=0.4$	-0.131	0.692	24.2%	0.211	0.812	11.7%	-0.180	0.562	11.2%
	$\rho_0=-0.3$	0.273	0.912	33.4%	-0.015	0.705	14.1%	-0.439	0.610	2.6%
	$\beta_{10}=0.8$	0.894	447.475	20.2%	-0.492	1.930	9.8%	0.415	1.367	8.9%
	$\beta_{20}=0.2$	-0.001	0.115	6.5%	-0.001	0.114	5.4%	0.000	0.113	5.2%
	$\beta_{30}=1.5$	-0.002	0.099	6.3%	0.003	0.099	5.5%	0.001	0.098	5.3%
	$\lambda_0=0.4$	-0.116	0.693	25.7%	0.261	0.852	13.3%	-0.155	0.552	11.1%
	$\rho_0=-0.2$	0.228	0.900	34.4%	-0.108	0.719	14.0%	-0.515	0.677	4.2%
	$\beta_{10}=0.8$	1.917	457.908	21.2%	-0.601	2.046	11.0%	0.366	1.349	9.3%
	$\beta_{20}=0.2$	-0.002	0.115	6.8%	-0.002	0.115	5.9%	-0.001	0.115	5.8%
	$\beta_{30}=1.5$	-0.003	0.098	6.2%	0.002	0.098	5.2%	-0.001	0.097	5.2%
	$\lambda_0=0.4$	-0.117	0.696	25.6%	0.284	0.855	13.9%	-0.132	0.546	11.2%
	$\rho_0=-0.1$	0.209	0.883	36.7%	-0.169	0.744	15.2%	-0.591	0.743	6.3%
	$\beta_{10}=0.8$	0.697	480.756	21.3%	-0.668	2.061	12.6%	0.305	1.327	9.3%
	$\beta_{20}=0.2$	0.000	0.115	6.9%	0.000	0.114	5.9%	0.000	0.114	5.7%
	$\beta_{30}=1.5$	-0.005	0.099	6.2%	0.001	0.098	5.2%	-0.003	0.098	5.2%

Table 2.A3: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 200$, $\lambda_0 < 0$, $\rho_0 = -0.4$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 10$	$\lambda_0 = -0.9$	0.046	0.171	4.5%	-0.002	0.247	5.6%	0.069	0.181	4.9%
	$\rho_0 = -0.4$	-0.055	0.269	5.9%	-0.031	0.264	3.5%	-0.129	0.275	6.1%
	$\beta_{10} = 0.8$	-0.032	0.375	5.5%	0.007	0.400	5.4%	-0.049	0.372	5.1%
	$\beta_{20} = 0.2$	-0.001	0.116	6.9%	-0.002	0.116	5.6%	-0.001	0.114	6.2%
	$\beta_{30} = 1.5$	-0.003	0.100	6.2%	-0.006	0.100	5.6%	-0.004	0.099	5.4%
	$\lambda_0 = -0.7$	-0.002	0.209	5.0%	0.000	0.238	5.4%	0.031	0.203	5.8%
	$\rho_0 = -0.4$	-0.019	0.291	9.0%	-0.023	0.266	3.9%	-0.099	0.283	7.2%
	$\beta_{10} = 0.8$	0.003	0.396	6.7%	0.000	0.407	5.7%	-0.025	0.388	5.9%
	$\beta_{20} = 0.2$	0.000	0.116	7.2%	0.000	0.116	5.8%	0.000	0.114	6.4%
	$\beta_{30} = 1.5$	-0.005	0.099	6.4%	-0.004	0.099	5.7%	-0.005	0.098	5.6%
	$\lambda_0 = -0.3$	-0.026	0.227	9.7%	-0.001	0.213	5.6%	0.009	0.200	7.8%
	$\rho_0 = -0.4$	-0.006	0.310	11.4%	-0.008	0.269	4.8%	-0.085	0.286	8.3%
	$\beta_{10} = 0.8$	0.280	25.189	7.1%	0.001	0.418	5.1%	-0.008	0.406	5.7%
	$\beta_{20} = 0.2$	0.000	0.114	7.1%	0.000	0.114	5.4%	0.000	0.112	6.2%
	$\beta_{30} = 1.5$	-0.007	0.101	6.2%	-0.004	0.099	5.3%	-0.006	0.099	5.4%
	$\lambda_0 = -0.2$	-0.025	0.221	9.5%	0.000	0.202	5.5%	0.008	0.191	7.8%
	$\rho_0 = -0.4$	-0.010	0.311	11.4%	-0.008	0.269	5.1%	-0.086	0.286	8.7%
	$\beta_{10} = 0.8$	0.036	38.752	7.4%	0.003	0.432	5.6%	-0.004	0.416	6.3%
	$\beta_{20} = 0.2$	-0.002	0.116	7.4%	-0.001	0.117	6.0%	-0.002	0.114	6.4%
	$\beta_{30} = 1.5$	-0.006	0.101	6.5%	-0.003	0.099	5.5%	-0.005	0.099	5.5%
	$\lambda_0 = -0.1$	-0.026	0.218	9.6%	0.001	0.193	5.6%	0.006	0.184	7.6%
	$\rho_0 = -0.4$	-0.011	0.311	11.4%	-0.007	0.270	5.1%	-0.086	0.285	8.3%
	$\beta_{10} = 0.8$	-0.093	12.998	7.7%	0.000	0.430	5.3%	-0.005	0.417	6.2%
	$\beta_{20} = 0.2$	-0.001	0.116	7.1%	0.000	0.116	5.9%	-0.001	0.114	6.3%
	$\beta_{30} = 1.5$	-0.006	0.102	6.8%	-0.003	0.099	5.4%	-0.005	0.099	5.6%
$J = 20$	$\lambda_0 = -0.9$	0.080	0.230	4.7%	0.013	0.379	6.0%	0.097	0.227	4.3%
	$\rho_0 = -0.4$	-0.052	0.381	4.1%	-0.031	0.385	5.1%	-0.222	0.374	3.0%
	$\beta_{10} = 0.8$	-0.077	60.727	4.6%	-0.010	0.450	5.4%	-0.073	0.386	4.2%
	$\beta_{20} = 0.2$	0.001	0.116	6.9%	0.000	0.114	5.3%	0.001	0.114	5.7%
	$\beta_{30} = 1.5$	0.002	0.100	6.1%	-0.002	0.099	5.2%	0.001	0.098	5.2%
	$\lambda_0 = -0.7$	0.009	0.268	5.1%	0.019	0.360	5.7%	0.040	0.248	4.3%
	$\rho_0 = -0.4$	-0.012	0.403	6.6%	-0.025	0.396	5.8%	-0.185	0.373	3.1%
	$\beta_{10} = 0.8$	0.380	76.503	4.8%	-0.012	0.460	5.0%	-0.029	0.403	4.5%
	$\beta_{20} = 0.2$	-0.001	0.115	6.9%	-0.001	0.115	5.5%	-0.001	0.113	5.9%
	$\beta_{30} = 1.5$	0.000	0.099	6.0%	-0.001	0.099	5.3%	0.001	0.098	5.2%
	$\lambda_0 = -0.3$	-0.053	0.330	7.1%	0.013	0.317	5.9%	-0.008	0.269	6.6%
	$\rho_0 = -0.4$	0.029	0.448	11.5%	-0.006	0.398	5.5%	-0.148	0.380	3.4%
	$\beta_{10} = 0.8$	-0.190	101.158	6.3%	-0.007	0.488	5.0%	0.015	0.455	5.5%
	$\beta_{20} = 0.2$	-0.003	0.116	7.1%	-0.003	0.115	6.0%	-0.002	0.114	6.4%
	$\beta_{30} = 1.5$	-0.008	0.101	6.7%	-0.004	0.099	5.7%	-0.004	0.099	5.7%
	$\lambda_0 = -0.2$	-0.059	0.335	10.2%	0.016	0.303	6.6%	-0.017	0.268	7.3%
	$\rho_0 = -0.4$	0.022	0.452	10.9%	-0.007	0.402	5.5%	-0.149	0.382	3.7%
	$\beta_{10} = 0.8$	0.795	116.361	7.2%	-0.009	0.506	5.6%	0.029	0.475	6.0%
	$\beta_{20} = 0.2$	-0.003	0.116	6.9%	-0.003	0.115	5.5%	-0.003	0.115	6.1%
	$\beta_{30} = 1.5$	-0.008	0.100	6.2%	-0.004	0.098	5.2%	-0.004	0.098	5.2%
	$\lambda_0 = -0.1$	-0.075	0.338	11.3%	0.004	0.288	6.4%	-0.029	0.262	6.7%
	$\rho_0 = -0.4$	0.037	0.455	11.1%	0.002	0.399	5.6%	-0.135	0.379	3.6%
	$\beta_{10} = 0.8$	-0.338	130.452	7.1%	-0.005	0.503	4.8%	0.037	0.485	5.6%
	$\beta_{20} = 0.2$	0.000	0.116	7.1%	0.001	0.115	5.8%	0.000	0.115	6.4%
	$\beta_{30} = 1.5$	-0.006	0.100	6.3%	-0.003	0.098	5.1%	-0.003	0.098	5.2%
$J = 100$	$\lambda_0 = -0.9$	0.247	0.603	8.8%	0.161	1.244	7.3%	0.161	0.402	6.5%
	$\rho_0 = -0.4$	0.237	0.823	27.3%	0.171	0.732	20.2%	-0.462	0.542	2.6%
	$\beta_{10} = 0.8$	-8.396	331.362	5.6%	-0.116	0.992	6.2%	-0.117	0.460	3.7%
	$\beta_{20} = 0.2$	0.000	0.115	6.7%	-0.002	0.115	5.9%	0.000	0.114	5.7%
	$\beta_{30} = 1.5$	0.004	0.100	6.1%	0.000	0.100	5.2%	0.002	0.098	5.2%
	$\lambda_0 = -0.7$	0.149	0.614	9.9%	0.202	1.183	7.8%	0.063	0.432	7.3%
	$\rho_0 = -0.4$	0.294	0.865	30.7%	0.176	0.730	19.4%	-0.434	0.536	2.2%
	$\beta_{10} = 0.8$	-0.231	366.654	6.6%	-0.160	1.045	6.0%	-0.050	0.499	4.3%
	$\beta_{20} = 0.2$	0.000	0.115	6.3%	-0.001	0.115	5.6%	-0.001	0.114	5.5%
	$\beta_{30} = 1.5$	0.002	0.100	5.9%	0.001	0.100	5.3%	0.001	0.099	5.3%
	$\lambda_0 = -0.3$	0.011	0.679	13.6%	0.239	1.089	8.4%	-0.100	0.508	6.6%
	$\rho_0 = -0.4$	0.321	0.904	33.5%	0.155	0.735	18.7%	-0.406	0.538	2.0%
	$\beta_{10} = 0.8$	-3.462	401.156	10.0%	-0.256	1.235	7.1%	0.108	0.653	4.6%
	$\beta_{20} = 0.2$	-0.001	0.115	6.3%	-0.001	0.114	5.7%	-0.001	0.114	5.4%
	$\beta_{30} = 1.5$	-0.002	0.099	6.2%	-0.001	0.099	5.3%	-0.001	0.099	5.4%
	$\lambda_0 = -0.2$	-0.040	0.686	14.0%	0.202	1.029	8.0%	-0.132	0.527	7.2%
	$\rho_0 = -0.4$	0.343	0.917	34.7%	0.156	0.727	17.9%	-0.398	0.538	1.7%
	$\beta_{10} = 0.8$	1.789	413.325	10.4%	-0.237	1.253	6.6%	0.154	0.709	4.8%
	$\beta_{20} = 0.2$	0.000	0.115	6.8%	0.000	0.113	5.5%	0.000	0.113	5.4%
	$\beta_{30} = 1.5$	-0.003	0.099	6.1%	-0.001	0.099	5.4%	-0.002	0.099	5.4%
	$\lambda_0 = -0.1$	-0.058	0.700	15.4%	0.208	0.997	8.4%	-0.151	0.543	8.3%
	$\rho_0 = -0.4$	0.337	0.923	34.8%	0.127	0.723	17.3%	-0.389	0.541	1.8%
	$\beta_{10} = 0.8$	7.192	413.014	11.7%	-0.258	1.338	6.9%	0.194	0.783	5.7%
	$\beta_{20} = 0.2$	0.000	0.116	6.9%	-0.002	0.115	6.1%	0.000	0.115	5.9%
	$\beta_{30} = 1.5$	-0.002	0.100	6.8%	0.000	0.101	5.7%	-0.001	0.100	5.7%

Table 2.A4: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 = 0.4$, $\rho_0 \geq 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 50$	$\lambda_0 = 0.4$	-0.011	0.271	14.3%	0.490	0.558	68.3%	0.177	0.270	22.5%
	$\rho_0 = 0.9$	-0.041	0.142	20.5%	-0.383	0.478	41.7%	-0.130	0.189	8.7%
	$\beta_{10} = 0.8$	-0.021	31.658	12.2%	-1.141	1.328	65.9%	-0.415	0.755	21.7%
	$\beta_{20} = 0.2$	-0.001	0.050	5.3%	-0.001	0.051	4.8%	-0.000	0.050	4.3%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.002	0.044	4.0%	0.005	0.043	3.6%
	$\lambda_0 = 0.4$	-0.043	0.278	15.6%	0.209	0.353	32.0%	0.073	0.213	19.9%
	$\rho_0 = 0.7$	-0.039	0.201	17.7%	-0.210	0.348	19.7%	-0.128	0.220	9.3%
	$\beta_{10} = 0.8$	-0.037	33.643	11.0%	-0.485	0.846	29.1%	-0.169	0.545	16.5%
	$\beta_{20} = 0.2$	-0.001	0.050	5.3%	-0.001	0.051	4.7%	-0.001	0.050	4.7%
	$\beta_{30} = 1.5$	-0.003	0.043	4.8%	-0.004	0.043	4.0%	0.001	0.043	4.0%
	$\lambda_0 = 0.4$	-0.052	0.241	11.3%	0.033	0.193	11.5%	0.005	0.175	11.1%
	$\rho_0 = 0.3$	-0.016	0.261	12.7%	-0.069	0.286	9.5%	-0.103	0.250	8.2%
	$\beta_{10} = 0.8$	0.258	23.443	9.2%	-0.075	0.482	9.4%	-0.009	0.443	8.6%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.003	0.043	4.5%	-0.002	0.043	4.2%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.045	0.213	9.4%	0.016	0.163	9.1%	-0.008	0.158	9.4%
	$\rho_0 = 0.1$	-0.010	0.279	10.2%	-0.051	0.287	7.5%	-0.088	0.260	8.1%
	$\beta_{10} = 0.8$	0.024	19.059	7.6%	-0.036	0.413	7.0%	0.021	0.403	7.3%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.5%	-0.002	0.043	4.2%	-0.001	0.043	4.5%
	$\lambda_0 = 0.4$	-0.041	0.199	8.5%	0.011	0.150	7.9%	-0.012	0.147	8.6%
	$\rho_0 = 0$	-0.009	0.285	9.2%	-0.043	0.288	6.9%	-0.080	0.262	8.0%
	$\beta_{10} = 0.8$	0.003	12.653	6.8%	-0.024	0.387	6.3%	0.031	0.380	6.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.5%	-0.002	0.043	4.3%	-0.001	0.043	4.5%
$J = 100$	$\lambda_0 = 0.4$	-0.029	0.356	16.1%	0.556	0.612	71.3%	0.227	0.319	33.8%
	$\rho_0 = 0.9$	-0.060	0.214	31.8%	-0.496	0.617	41.4%	-0.219	0.304	13.3%
	$\beta_{10} = 0.8$	3.435	115.738	12.8%	-1.298	1.448	69.6%	-0.530	0.843	32.1%
	$\beta_{20} = 0.2$	-0.001	0.050	5.1%	-0.001	0.051	4.7%	-0.000	0.050	4.1%
	$\beta_{30} = 1.5$	-0.002	0.043	4.6%	-0.000	0.043	3.9%	0.003	0.043	3.5%
	$\lambda_0 = 0.4$	-0.082	0.397	19.1%	0.340	0.478	41.8%	0.096	0.261	24.4%
	$\rho_0 = 0.7$	-0.046	0.294	26.5%	-0.338	0.527	26.6%	-0.215	0.334	11.2%
	$\beta_{10} = 0.8$	3.710	123.538	12.4%	-0.791	1.128	39.4%	-0.224	0.646	21.0%
	$\beta_{20} = 0.2$	-0.001	0.050	5.1%	-0.001	0.051	4.7%	-0.001	0.050	4.4%
	$\beta_{30} = 1.5$	-0.003	0.043	4.7%	-0.001	0.043	3.8%	0.000	0.043	4.0%
	$\lambda_0 = 0.4$	-0.091	0.370	15.9%	0.091	0.291	16.4%	-0.001	0.240	13.3%
	$\rho_0 = 0.3$	-0.017	0.392	18.8%	-0.134	0.442	14.8%	-0.188	0.371	9.9%
	$\beta_{10} = 0.8$	2.018	105.694	12.4%	-0.211	0.699	14.1%	0.004	0.586	11.4%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.002	0.043	4.1%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.087	0.348	13.9%	0.039	0.244	10.0%	-0.021	0.231	11.5%
	$\rho_0 = 0.1$	-0.003	0.421	16.0%	-0.068	0.420	10.0%	-0.171	0.384	9.8%
	$\beta_{10} = 0.8$	1.499	89.263	11.0%	-0.089	0.592	8.4%	0.051	0.567	9.6%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.1%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.084	0.336	12.9%	0.021	0.228	7.8%	-0.027	0.225	10.5%
	$\rho_0 = 0$	0.003	0.432	14.8%	-0.039	0.414	8.5%	-0.163	0.388	9.4%
	$\beta_{10} = 0.8$	0.694	82.839	10.3%	-0.047	0.555	6.6%	0.064	0.551	8.7%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.6%	-0.001	0.042	4.0%	-0.001	0.043	4.4%
$J = 500$	$\lambda_0 = 0.4$	0.156	0.595	18.3%	0.775	1.077	43.6%	0.309	0.520	17.0%
	$\rho_0 = 0.9$	-0.104	0.482	48.2%	-0.764	1.023	34.4%	-1.099	1.233	24.3%
	$\beta_{10} = 0.8$	4.637	252.411	16.2%	-1.806	2.599	43.6%	-0.727	1.278	23.3%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.4%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	0.000	0.043	4.6%	0.001	0.043	4.2%	0.000	0.043	4.3%
	$\lambda_0 = 0.4$	0.060	0.638	22.8%	0.637	1.024	34.1%	0.199	0.503	15.5%
	$\rho_0 = 0.7$	-0.028	0.603	48.8%	-0.665	0.974	26.2%	-1.076	1.204	22.7%
	$\beta_{10} = 0.8$	1.603	250.286	20.5%	-1.490	2.394	33.8%	-0.465	1.190	14.9%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.5%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	-0.000	0.043	4.6%	0.001	0.043	4.4%	0.000	0.043	4.3%
	$\lambda_0 = 0.4$	-0.030	0.679	27.4%	0.392	0.986	18.8%	0.008	0.499	11.9%
	$\rho_0 = 0.3$	0.080	0.793	42.9%	-0.314	0.784	20.5%	-0.884	1.009	16.2%
	$\beta_{10} = 0.8$	1.514	232.446	25.5%	-0.916	2.286	18.0%	-0.017	1.178	11.4%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.050	5.1%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	0.002	0.043	4.5%	-0.000	0.043	4.4%
	$\lambda_0 = 0.4$	-0.058	0.687	27.7%	0.245	0.905	12.1%	-0.067	0.510	10.8%
	$\rho_0 = 0.1$	0.143	0.849	39.5%	-0.062	0.707	22.8%	-0.750	0.880	11.5%
	$\beta_{10} = 0.8$	-0.130	220.392	25.8%	-0.575	2.107	11.6%	0.157	1.202	10.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.000	0.051	5.1%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	0.001	0.043	4.4%	-0.000	0.043	4.2%
	$\lambda_0 = 0.4$	-0.077	0.693	27.2%	0.206	0.919	10.2%	-0.103	0.517	10.7%
	$\rho_0 = 0$	0.187	0.871	38.4%	0.054	0.701	23.6%	-0.677	0.811	8.7%
	$\beta_{10} = 0.8$	-0.004	214.543	25.5%	-0.483	2.140	10.3%	0.241	1.218	10.5%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.000	0.050	5.1%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	0.001	0.043	4.6%	-0.000	0.043	4.2%

Table 2.A5: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 = -0.4$, $\rho_0 \geq 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 50$	$\lambda_0 = -0.4$	0.008	0.306	7.0%	0.922	1.045	63.7%	0.165	0.378	5.8%
	$\rho_0 = 0.9$	-0.014	0.082	13.5%	-0.469	0.637	39.8%	-0.069	0.134	2.6%
	$\beta_{10} = 0.8$	-0.333	25.612	7.2%	-0.930	1.088	61.3%	-0.172	0.605	10.1%
	$\beta_{20} = 0.2$	-0.000	0.050	5.6%	0.002	0.052	4.6%	0.001	0.050	5.3%
	$\beta_{30} = 1.5$	-0.001	0.044	5.0%	0.013	0.047	5.3%	0.006	0.045	4.6%
	$\lambda_0 = -0.4$	0.001	0.321	9.2%	0.350	0.563	30.0%	0.138	0.350	11.3%
	$\rho_0 = 0.7$	-0.030	0.161	12.1%	-0.235	0.353	12.8%	-0.117	0.212	6.3%
	$\beta_{10} = 0.8$	-0.502	28.624	6.8%	-0.353	0.600	25.9%	-0.141	0.413	9.6%
	$\beta_{20} = 0.2$	-0.000	0.051	5.5%	0.000	0.051	5.1%	0.000	0.051	5.3%
	$\beta_{30} = 1.5$	-0.001	0.044	5.0%	0.003	0.045	4.3%	0.004	0.044	4.6%
	$\lambda_0 = -0.4$	-0.021	0.295	8.3%	0.074	0.315	11.7%	0.057	0.278	12.1%
	$\rho_0 = 0.3$	-0.028	0.247	12.2%	-0.129	0.271	6.8%	-0.124	0.267	9.2%
	$\beta_{10} = 0.8$	-0.261	21.705	7.0%	-0.074	0.356	8.7%	-0.057	0.323	8.7%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.002	0.044	5.0%	-0.001	0.044	4.6%	0.000	0.043	4.8%
	$\lambda_0 = -0.4$	-0.027	0.277	7.6%	0.040	0.275	9.0%	0.036	0.253	11.2%
	$\rho_0 = 0.1$	-0.023	0.274	11.6%	-0.101	0.277	6.1%	-0.120	0.284	9.2%
	$\beta_{10} = 0.8$	-0.280	17.745	6.4%	-0.040	0.317	6.7%	-0.036	0.298	7.8%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.5%
	$\beta_{30} = 1.5$	-0.002	0.043	5.0%	-0.001	0.043	4.6%	-0.001	0.043	4.6%
	$\lambda_0 = -0.4$	-0.029	0.267	7.5%	0.028	0.259	7.9%	0.024	0.237	10.3%
	$\rho_0 = 0$	-0.020	0.284	11.1%	-0.086	0.278	5.7%	-0.111	0.284	9.1%
	$\beta_{10} = 0.8$	-0.180	14.784	6.0%	-0.028	0.303	6.0%	-0.024	0.283	6.8%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.4%	-0.000	0.051	5.5%
	$\beta_{30} = 1.5$	-0.002	0.043	4.9%	-0.001	0.043	4.6%	-0.001	0.043	4.6%
$J = 100$	$\lambda_0 = -0.4$	0.031	0.423	5.6%	1.097	1.196	75.6%	0.334	0.540	13.4%
	$\rho_0 = 0.9$	-0.037	0.172	24.8%	-0.701	0.841	44.5%	-0.169	0.267	5.0%
	$\beta_{10} = 0.8$	-0.289	95.621	9.2%	-1.102	1.225	73.7%	-0.340	0.690	16.5%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	0.001	0.051	5.0%	0.001	0.051	4.9%
	$\beta_{30} = 1.5$	-0.001	0.044	4.8%	0.009	0.045	5.1%	0.005	0.044	4.2%
	$\lambda_0 = -0.4$	0.009	0.429	8.6%	0.586	0.806	43.6%	0.250	0.466	18.2%
	$\rho_0 = 0.7$	-0.050	0.274	21.0%	-0.439	0.597	20.7%	-0.242	0.372	10.2%
	$\beta_{10} = 0.8$	-1.914	116.485	8.3%	-0.588	0.829	39.9%	-0.252	0.511	15.7%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	0.001	0.051	5.2%	0.000	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.044	4.8%	0.004	0.044	4.5%	0.003	0.043	4.3%
	$\lambda_0 = -0.4$	-0.024	0.394	9.2%	0.175	0.460	16.4%	0.111	0.358	15.9%
	$\rho_0 = 0.3$	-0.035	0.383	18.5%	-0.222	0.447	11.1%	-0.251	0.428	12.6%
	$\beta_{10} = 0.8$	-0.743	99.351	7.7%	-0.175	0.487	13.7%	-0.111	0.395	12.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.4%	-0.000	0.051	5.5%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	0.000	0.043	4.5%	0.000	0.043	4.5%
	$\lambda_0 = -0.4$	-0.032	0.372	8.8%	0.086	0.390	10.3%	0.067	0.325	13.7%
	$\rho_0 = 0.1$	-0.024	0.415	17.0%	-0.143	0.412	7.7%	-0.236	0.430	11.8%
	$\beta_{10} = 0.8$	0.053	81.565	6.9%	-0.086	0.420	8.0%	-0.067	0.363	10.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.000	0.043	4.6%	-0.000	0.043	4.6%
	$\lambda_0 = -0.4$	-0.035	0.362	8.5%	0.054	0.368	7.9%	0.047	0.307	11.7%
	$\rho_0 = 0$	-0.018	0.426	16.3%	-0.106	0.401	6.5%	-0.222	0.424	10.9%
	$\beta_{10} = 0.8$	-0.326	74.845	6.7%	-0.055	0.399	6.2%	-0.047	0.344	8.6%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.6%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.000	0.043	4.5%	-0.001	0.043	4.6%
$J = 500$	$\lambda_0 = -0.4$	0.392	0.835	20.7%	1.118	1.549	42.0%	0.669	0.890	36.4%
	$\rho_0 = 0.9$	-0.151	0.564	64.2%	-0.804	1.057	27.8%	-1.256	1.385	29.7%
	$\beta_{10} = 0.8$	6.607	239.891	13.0%	-1.114	1.603	42.8%	-0.670	0.962	39.3%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	-0.000	0.051	5.7%	0.000	0.051	5.4%
	$\beta_{30} = 1.5$	0.001	0.043	4.9%	0.003	0.044	5.1%	0.002	0.043	4.7%
	$\lambda_0 = -0.4$	0.270	0.796	19.7%	0.889	1.430	31.4%	0.471	0.756	24.8%
	$\rho_0 = 0.7$	-0.057	0.639	59.8%	-0.627	0.939	23.6%	-1.155	1.280	25.5%
	$\beta_{10} = 0.8$	2.694	239.377	14.0%	-0.888	1.454	30.0%	-0.472	0.782	23.9%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	0.000	0.051	5.4%	0.000	0.051	5.3%
	$\beta_{30} = 1.5$	0.001	0.043	4.9%	0.002	0.043	4.9%	0.001	0.043	4.6%
	$\lambda_0 = -0.4$	0.154	0.757	18.9%	0.480	1.258	15.6%	0.207	0.588	12.3%
	$\rho_0 = 0.3$	0.104	0.772	49.8%	-0.238	0.745	24.3%	-0.925	1.040	17.8%
	$\beta_{10} = 0.8$	0.742	222.633	15.2%	-0.480	1.274	15.0%	-0.207	0.612	11.5%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	0.000	0.050	5.3%	0.000	0.051	5.3%
	$\beta_{30} = 1.5$	0.000	0.043	4.8%	0.001	0.043	4.8%	0.001	0.043	4.6%
	$\lambda_0 = -0.4$	0.123	0.735	17.7%	0.268	1.173	9.3%	0.109	0.535	9.3%
	$\rho_0 = 0.1$	0.172	0.828	45.1%	0.007	0.678	26.0%	-0.792	0.903	12.8%
	$\beta_{10} = 0.8$	-0.645	210.096	14.5%	-0.268	1.189	9.1%	-0.109	0.560	8.5%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.050	5.2%	0.000	0.051	5.3%
	$\beta_{30} = 1.5$	0.000	0.043	4.8%	0.001	0.044	5.2%	0.000	0.043	4.6%
	$\lambda_0 = -0.4$	0.107	0.722	16.7%	0.169	1.191	7.9%	0.064	0.515	8.2%
	$\rho_0 = 0$	0.212	0.849	43.0%	0.121	0.686	27.8%	-0.719	0.831	9.9%
	$\beta_{10} = 0.8$	-0.610	204.345	13.7%	-0.167	1.209	7.3%	-0.064	0.540	7.2%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.050	4.9%	0.000	0.051	5.3%
	$\beta_{30} = 1.5$	0.000	0.043	4.8%	0.000	0.043	4.7%	0.000	0.043	4.6%

Table 2.A6: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 = 0.4$, $\rho_0 < 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 50$	$\lambda_0 = 0.4$	-0.023	0.092	2.9%	-0.021	0.101	3.1%	-0.113	0.284	2.5%
	$\rho_0 = -0.9$	0.071	0.217	2.8%	0.148	0.253	4.3%	0.136	0.409	1.8%
	$\beta_{10} = 0.8$	0.056	0.264	3.8%	0.050	0.282	4.0%	0.264	0.681	3.6%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.2%	-0.000	0.051	5.1%
	$\beta_{30} = 1.5$	0.000	0.043	4.5%	0.001	0.043	4.7%	0.003	0.044	4.4%
	$\lambda_0 = 0.4$	-0.020	0.107	4.0%	-0.011	0.104	3.4%	-0.060	0.212	2.9%
	$\rho_0 = -0.7$	0.017	0.257	2.9%	0.063	0.245	3.4%	0.019	0.330	1.8%
	$\beta_{10} = 0.8$	0.048	0.295	4.4%	0.028	0.288	3.9%	0.142	0.521	3.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.5%	-0.001	0.051	5.1%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	0.000	0.043	4.5%	0.000	0.043	4.5%
	$\lambda_0 = 0.4$	-0.028	0.156	6.7%	0.003	0.124	6.2%	-0.015	0.138	7.1%
	$\rho_0 = -0.3$	-0.008	0.294	7.1%	-0.025	0.281	5.2%	-0.077	0.285	7.5%
	$\beta_{10} = 0.8$	-0.084	10.707	5.6%	-0.005	0.330	5.1%	0.036	0.357	5.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.0%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	-0.001	0.043	4.3%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.031	0.167	7.1%	0.006	0.132	6.8%	-0.012	0.139	8.0%
	$\rho_0 = -0.2$	-0.009	0.292	7.5%	-0.034	0.286	6.0%	-0.082	0.281	7.7%
	$\beta_{10} = 0.8$	0.057	10.971	5.9%	-0.011	0.346	5.5%	0.030	0.362	6.3%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.0%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	-0.001	0.043	4.3%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.036	0.184	7.8%	0.009	0.141	7.5%	-0.011	0.145	8.5%
	$\rho_0 = -0.1$	-0.009	0.290	8.3%	-0.040	0.289	6.6%	-0.086	0.276	8.0%
	$\beta_{10} = 0.8$	0.116	15.824	6.4%	-0.018	0.365	6.0%	0.027	0.374	6.7%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.0%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.002	0.043	4.5%	-0.001	0.043	4.3%	-0.001	0.043	4.4%
$J = 100$	$\lambda_0 = 0.4$	-0.053	0.192	3.9%	-0.018	0.163	3.5%	-0.176	0.372	2.9%
	$\rho_0 = -0.9$	0.142	0.356	4.7%	0.196	0.374	5.6%	0.189	0.493	2.2%
	$\beta_{10} = 0.8$	0.137	31.221	3.9%	0.044	0.409	3.6%	0.411	0.883	3.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.2%	-0.000	0.051	4.8%
	$\beta_{30} = 1.5$	-0.000	0.043	4.6%	0.000	0.043	4.6%	0.001	0.043	4.1%
	$\lambda_0 = 0.4$	-0.054	0.224	5.4%	-0.011	0.171	4.2%	-0.119	0.304	3.1%
	$\rho_0 = -0.7$	0.077	0.394	5.6%	0.109	0.375	5.2%	0.041	0.415	1.9%
	$\beta_{10} = 0.8$	-0.227	41.646	5.2%	0.027	0.426	4.1%	0.278	0.729	3.4%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.1%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.001	0.043	4.3%	0.000	0.043	4.2%
	$\lambda_0 = 0.4$	-0.069	0.289	9.7%	0.008	0.196	6.4%	-0.046	0.227	6.9%
	$\rho_0 = -0.3$	0.019	0.444	8.1%	-0.010	0.412	5.1%	-0.131	0.397	3.6%
	$\beta_{10} = 0.8$	1.206	65.656	8.1%	-0.016	0.484	5.7%	0.109	0.555	6.2%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.2%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.075	0.305	10.7%	0.015	0.206	7.6%	-0.038	0.225	8.3%
	$\rho_0 = -0.2$	0.013	0.444	10.3%	-0.033	0.420	5.7%	-0.148	0.400	6.5%
	$\beta_{10} = 0.8$	0.995	69.403	8.8%	-0.033	0.506	6.4%	0.090	0.550	7.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.1%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.2%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.079	0.318	11.7%	0.023	0.217	8.8%	-0.031	0.224	9.8%
	$\rho_0 = -0.1$	0.007	0.439	13.1%	-0.052	0.429	7.8%	-0.159	0.396	8.4%
	$\beta_{10} = 0.8$	0.411	76.428	9.5%	-0.053	0.531	7.3%	0.075	0.550	8.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.2%	-0.001	0.043	4.3%
$J = 500$	$\lambda_0 = 0.4$	-0.178	0.661	18.0%	0.034	0.822	4.8%	-0.319	0.641	14.5%
	$\rho_0 = -0.9$	0.584	1.018	26.1%	0.446	0.808	13.7%	0.066	0.378	1.0%
	$\beta_{10} = 0.8$	-0.794	155.654	17.3%	-0.078	1.921	4.8%	0.745	1.508	13.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.2%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.001	0.043	4.7%	-0.001	0.043	4.2%
	$\lambda_0 = 0.4$	-0.152	0.667	20.2%	0.092	0.815	6.1%	-0.274	0.606	13.4%
	$\rho_0 = -0.7$	0.483	0.985	28.7%	0.297	0.755	14.2%	-0.116	0.397	0.8%
	$\beta_{10} = 0.8$	-0.658	169.215	18.9%	-0.215	1.903	5.9%	0.639	1.425	12.9%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.050	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.001	0.043	4.5%	-0.001	0.043	4.1%
	$\lambda_0 = 0.4$	-0.114	0.680	23.7%	0.221	0.819	10.0%	-0.177	0.550	11.6%
	$\rho_0 = -0.3$	0.313	0.931	34.8%	-0.015	0.724	15.3%	-0.455	0.614	2.8%
	$\beta_{10} = 0.8$	0.963	196.273	22.1%	-0.515	1.913	9.5%	0.414	1.295	11.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.051	4.9%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.000	0.043	4.5%	-0.001	0.043	4.2%
	$\lambda_0 = 0.4$	-0.104	0.684	24.7%	0.263	0.844	11.8%	-0.151	0.539	11.4%
	$\rho_0 = -0.2$	0.271	0.913	36.0%	-0.094	0.734	15.3%	-0.531	0.681	4.5%
	$\beta_{10} = 0.8$	0.360	202.638	23.0%	-0.614	1.968	11.3%	0.353	1.269	10.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.000	0.051	5.1%	-0.001	0.051	4.9%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.000	0.043	4.5%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.093	0.687	25.8%	0.307	0.863	13.5%	-0.127	0.534	11.3%
	$\rho_0 = -0.1$	0.233	0.895	37.6%	-0.175	0.754	15.7%	-0.603	0.748	6.5%
	$\beta_{10} = 0.8$	-0.476	209.142	24.1%	-0.715	2.009	12.8%	0.298	1.260	10.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.050	5.0%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.000	0.043	4.5%	-0.001	0.043	4.3%

Table 2.A7: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 < 0$, $\rho_0 = -0.4$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 50$	$\lambda_0 = -0.9$	0.050	0.173	4.6%	-0.004	0.265	4.8%	0.068	0.179	5.4%
	$\rho_0 = -0.4$	-0.055	0.270	2.6%	-0.036	0.267	3.0%	-0.144	0.285	4.9%
	$\beta_{10} = 0.8$	-0.037	0.200	3.5%	0.003	0.249	4.7%	-0.050	0.203	3.9%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.000	0.043	4.9%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
	$\lambda_0 = -0.7$	-0.004	0.209	4.7%	-0.002	0.248	4.9%	0.024	0.201	5.8%
	$\rho_0 = -0.4$	-0.022	0.287	4.9%	-0.031	0.270	3.1%	-0.112	0.288	5.5%
	$\beta_{10} = 0.8$	0.003	0.231	4.0%	0.002	0.255	4.6%	-0.020	0.225	4.6%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.001	0.043	5.0%	-0.001	0.043	4.6%	-0.001	0.043	4.6%
	$\lambda_0 = -0.3$	-0.031	0.223	7.5%	-0.000	0.208	5.1%	0.001	0.194	7.2%
	$\rho_0 = -0.4$	-0.005	0.305	7.7%	-0.022	0.275	3.3%	-0.097	0.291	6.3%
	$\beta_{10} = 0.8$	-0.060	9.366	5.7%	0.000	0.271	4.5%	-0.001	0.258	5.4%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.001	0.043	5.0%	-0.001	0.043	4.8%	-0.001	0.043	4.7%
	$\lambda_0 = -0.2$	-0.032	0.217	7.4%	-0.000	0.197	5.1%	-0.002	0.185	7.2%
	$\rho_0 = -0.4$	-0.005	0.306	7.8%	-0.020	0.276	3.3%	-0.094	0.290	6.1%
	$\beta_{10} = 0.8$	-0.163	14.247	5.9%	0.000	0.275	4.5%	0.003	0.265	5.4%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.001	0.043	5.0%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
	$\lambda_0 = -0.1$	-0.033	0.210	7.3%	0.000	0.185	5.2%	-0.005	0.177	7.1%
	$\rho_0 = -0.4$	-0.004	0.306	7.6%	-0.019	0.276	3.4%	-0.091	0.290	6.1%
	$\beta_{10} = 0.8$	-0.130	20.040	5.9%	-0.000	0.280	4.5%	0.006	0.272	5.5%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.001	0.043	5.0%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
$J = 100$	$\lambda_0 = -0.9$	0.081	0.231	4.6%	0.015	0.393	5.7%	0.093	0.223	4.3%
	$\rho_0 = -0.4$	-0.047	0.376	3.7%	-0.031	0.393	5.2%	-0.234	0.377	2.1%
	$\beta_{10} = 0.8$	-0.324	27.040	3.6%	-0.010	0.327	4.6%	-0.069	0.225	3.5%
	$\beta_{20} = 0.2$	0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	0.000	0.043	4.8%	-0.000	0.043	4.6%	0.000	0.043	4.7%
	$\lambda_0 = -0.7$	0.011	0.268	5.0%	0.015	0.368	5.6%	0.034	0.243	4.3%
	$\rho_0 = -0.4$	-0.009	0.398	6.0%	-0.019	0.399	5.5%	-0.196	0.374	2.1%
	$\beta_{10} = 0.8$	-0.135	33.490	3.8%	-0.011	0.340	4.7%	-0.028	0.253	4.0%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.000	0.043	4.8%	-0.000	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.3$	-0.052	0.321	5.7%	0.014	0.311	5.9%	-0.013	0.260	6.5%
	$\rho_0 = -0.4$	0.024	0.437	10.1%	-0.003	0.406	5.7%	-0.165	0.380	2.5%
	$\beta_{10} = 0.8$	-0.708	46.302	4.8%	-0.015	0.367	4.8%	0.014	0.320	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.2$	-0.059	0.323	6.3%	0.013	0.294	5.9%	-0.021	0.258	6.5%
	$\rho_0 = -0.4$	0.026	0.441	10.2%	0.001	0.407	5.6%	-0.159	0.381	2.6%
	$\beta_{10} = 0.8$	-0.574	51.389	5.1%	-0.015	0.374	4.9%	0.024	0.338	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.1$	-0.063	0.323	7.8%	0.012	0.277	5.7%	-0.026	0.252	6.3%
	$\rho_0 = -0.4$	0.028	0.445	10.1%	0.004	0.407	5.5%	-0.153	0.382	2.6%
	$\beta_{10} = 0.8$	-0.337	54.842	6.0%	-0.015	0.383	4.9%	0.033	0.356	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.5%	-0.000	0.043	4.7%
$J = 500$	$\lambda_0 = -0.9$	0.260	0.616	8.7%	0.207	1.244	7.4%	0.168	0.407	6.8%
	$\rho_0 = -0.4$	0.299	0.856	30.1%	0.207	0.761	22.2%	-0.459	0.541	2.5%
	$\beta_{10} = 0.8$	0.814	152.743	6.7%	-0.154	0.934	6.8%	-0.124	0.337	5.8%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	0.000	0.051	5.6%	0.000	0.051	5.4%
	$\beta_{30} = 1.5$	0.001	0.043	4.9%	0.000	0.043	4.8%	0.000	0.043	4.7%
	$\lambda_0 = -0.7$	0.162	0.627	10.1%	0.217	1.186	7.5%	0.060	0.423	7.5%
	$\rho_0 = -0.4$	0.336	0.887	32.8%	0.197	0.756	21.6%	-0.443	0.537	2.3%
	$\beta_{10} = 0.8$	1.943	163.742	7.9%	-0.179	0.993	6.8%	-0.049	0.381	6.2%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	0.000	0.051	5.6%	0.000	0.051	5.3%
	$\beta_{30} = 1.5$	0.000	0.043	4.9%	0.000	0.043	4.7%	0.000	0.043	4.6%
	$\lambda_0 = -0.3$	0.017	0.681	13.6%	0.239	1.075	8.1%	-0.094	0.497	7.3%
	$\rho_0 = -0.4$	0.370	0.928	35.3%	0.164	0.748	19.6%	-0.417	0.539	1.9%
	$\beta_{10} = 0.8$	0.323	180.475	11.5%	-0.258	1.171	7.4%	0.102	0.558	6.5%
	$\beta_{20} = 0.2$	-0.000	0.051	5.6%	0.000	0.051	5.6%	-0.000	0.051	5.4%
	$\beta_{30} = 1.5$	-0.000	0.043	4.8%	0.001	0.043	4.8%	0.000	0.043	4.6%
	$\lambda_0 = -0.2$	-0.013	0.691	14.1%	0.232	1.049	8.3%	-0.122	0.514	7.9%
	$\rho_0 = -0.4$	0.379	0.938	36.1%	0.149	0.742	19.1%	-0.411	0.539	1.7%
	$\beta_{10} = 0.8$	-1.420	183.357	12.2%	-0.273	1.237	7.5%	0.142	0.620	7.1%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	0.000	0.051	5.5%	-0.000	0.051	5.4%
	$\beta_{30} = 1.5$	-0.000	0.043	4.9%	0.001	0.043	4.8%	-0.000	0.043	4.6%
	$\lambda_0 = -0.1$	-0.040	0.699	15.1%	0.233	1.010	8.5%	-0.145	0.529	8.7%
	$\rho_0 = -0.4$	0.381	0.943	36.2%	0.140	0.739	18.4%	-0.403	0.541	1.7%
	$\beta_{10} = 0.8$	0.302	185.535	13.3%	-0.297	1.299	7.6%	0.185	0.692	7.8%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	0.000	0.051	5.5%	-0.000	0.051	5.3%
	$\beta_{30} = 1.5$	-0.000	0.043	4.9%	0.000	0.043	4.8%	-0.000	0.043	4.6%

Table 2.A8: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 = 0.4$, $\rho_0 \geq 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 10$	$\lambda_0 = 0.4$	-0.001	0.126	7.0%	0.238	0.342	39.9%	0.070	0.161	5.4%
	$\rho_0 = 0.9$	-0.007	0.046	10.1%	-0.144	0.239	26.0%	-0.041	0.086	3.6%
	$\beta_{10} = 0.8$	0.020	2.195	6.3%	-0.547	0.896	34.8%	-0.167	0.612	7.3%
	$\beta_{20} = 0.2$	-0.001	0.049	5.4%	-0.003	0.066	5.3%	0.001	0.049	4.7%
	$\beta_{30} = 1.5$	-0.002	0.045	5.2%	-0.018	0.058	4.8%	0.009	0.046	4.2%
	$\lambda_0 = 0.4$	-0.009	0.124	9.0%	0.030	0.154	12.8%	0.021	0.118	10.2%
	$\rho_0 = 0.7$	-0.009	0.084	10.0%	-0.053	0.109	8.1%	-0.032	0.090	7.2%
	$\beta_{10} = 0.8$	0.023	0.374	6.6%	-0.064	0.440	9.2%	-0.048	0.357	7.6%
	$\beta_{20} = 0.2$	-0.001	0.050	5.4%	-0.002	0.058	5.0%	-0.001	0.049	5.2%
	$\beta_{30} = 1.5$	-0.003	0.045	5.0%	-0.009	0.048	4.3%	0.001	0.044	4.3%
	$\lambda_0 = 0.4$	-0.009	0.088	6.2%	0.004	0.085	6.2%	0.002	0.083	6.9%
	$\rho_0 = 0.3$	-0.004	0.110	6.6%	-0.016	0.109	5.1%	-0.021	0.110	6.4%
	$\beta_{10} = 0.8$	0.023	0.268	5.3%	-0.007	0.262	4.8%	-0.003	0.257	5.2%
	$\beta_{20} = 0.2$	-0.001	0.051	5.5%	-0.001	0.051	5.1%	-0.001	0.050	5.3%
	$\beta_{30} = 1.5$	-0.002	0.043	4.5%	-0.003	0.043	4.4%	-0.002	0.043	4.4%
	$\lambda_0 = 0.4$	-0.006	0.073	5.5%	0.002	0.070	5.5%	-0.001	0.068	5.2%
	$\rho_0 = 0.1$	-0.004	0.114	5.8%	-0.010	0.112	4.9%	-0.017	0.110	5.4%
	$\beta_{10} = 0.8$	0.017	0.235	5.1%	-0.003	0.230	4.7%	0.004	0.227	4.7%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.2%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.002	0.043	4.3%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.005	0.066	5.4%	0.002	0.065	5.2%	-0.002	0.062	5.2%
	$\rho_0 = 0$	-0.004	0.115	5.4%	-0.009	0.113	4.8%	-0.013	0.108	5.0%
	$\beta_{10} = 0.8$	0.014	0.222	5.1%	-0.001	0.220	4.7%	0.007	0.215	4.7%
	$\beta_{20} = 0.2$	-0.001	0.051	5.5%	-0.001	0.051	5.2%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.002	0.043	4.5%	-0.001	0.043	4.5%
$J = 20$	$\lambda_0 = 0.4$	-0.000	0.177	9.2%	0.356	0.446	55.8%	0.108	0.204	10.4%
	$\rho_0 = 0.9$	-0.017	0.078	13.0%	-0.247	0.339	35.7%	-0.066	0.114	4.8%
	$\beta_{10} = 0.8$	0.010	5.554	8.6%	-0.825	1.094	52.0%	-0.253	0.667	11.7%
	$\beta_{20} = 0.2$	-0.001	0.050	5.3%	-0.002	0.055	4.4%	0.000	0.050	4.4%
	$\beta_{30} = 1.5$	-0.002	0.044	4.8%	-0.008	0.048	4.3%	0.007	0.044	3.8%
	$\lambda_0 = 0.4$	-0.018	0.175	11.3%	0.072	0.213	19.0%	0.038	0.159	14.2%
	$\rho_0 = 0.7$	-0.018	0.121	12.3%	-0.096	0.176	10.3%	-0.060	0.133	8.5%
	$\beta_{10} = 0.8$	0.044	0.477	8.4%	-0.164	0.552	15.5%	-0.088	0.437	10.6%
	$\beta_{20} = 0.2$	-0.001	0.050	5.3%	-0.002	0.053	4.8%	-0.001	0.050	4.9%
	$\beta_{30} = 1.5$	-0.003	0.044	4.7%	-0.007	0.045	4.2%	0.001	0.043	4.1%
	$\lambda_0 = 0.4$	-0.021	0.133	7.6%	0.009	0.120	7.9%	0.003	0.115	8.4%
	$\rho_0 = 0.3$	-0.005	0.157	7.9%	-0.031	0.156	5.0%	-0.042	0.154	6.3%
	$\beta_{10} = 0.8$	0.051	0.356	6.1%	-0.018	0.330	5.9%	-0.005	0.320	6.3%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.050	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.5%	-0.003	0.043	4.3%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.015	0.108	6.1%	0.005	0.099	6.4%	-0.003	0.096	6.7%
	$\rho_0 = 0.1$	-0.005	0.163	6.1%	-0.022	0.160	4.9%	-0.034	0.156	5.9%
	$\beta_{10} = 0.8$	0.038	0.303	5.3%	-0.008	0.284	5.2%	0.008	0.278	5.2%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	-0.001	0.043	4.4%	-0.002	0.043	4.3%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.013	0.098	5.7%	0.003	0.092	5.9%	-0.005	0.088	6.0%
	$\rho_0 = 0$	-0.005	0.165	5.8%	-0.019	0.162	4.8%	-0.027	0.154	5.7%
	$\beta_{10} = 0.8$	0.033	0.282	5.0%	-0.005	0.268	4.8%	0.014	0.261	4.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.0%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	-0.002	0.043	4.3%	-0.001	0.043	4.4%
$J = 100$	$\lambda_0 = 0.4$	-0.028	0.356	16.1%	0.556	0.612	71.3%	0.227	0.319	33.8%
	$\rho_0 = 0.9$	-0.060	0.214	31.7%	-0.495	0.617	41.4%	-0.218	0.303	13.3%
	$\beta_{10} = 0.8$	3.503	115.793	12.7%	-1.298	1.448	69.5%	-0.529	0.843	32.1%
	$\beta_{20} = 0.2$	-0.001	0.050	5.1%	-0.001	0.051	4.7%	-0.000	0.050	4.1%
	$\beta_{30} = 1.5$	-0.002	0.043	4.6%	-0.000	0.043	3.9%	0.003	0.043	3.6%
	$\lambda_0 = 0.4$	-0.082	0.397	19.1%	0.339	0.478	41.7%	0.096	0.261	24.5%
	$\rho_0 = 0.7$	-0.046	0.294	26.5%	-0.338	0.527	26.5%	-0.215	0.333	11.2%
	$\beta_{10} = 0.8$	3.643	123.965	12.4%	-0.791	1.128	39.3%	-0.224	0.646	20.9%
	$\beta_{20} = 0.2$	-0.001	0.050	5.1%	-0.001	0.051	4.7%	-0.001	0.050	4.4%
	$\beta_{30} = 1.5$	-0.003	0.043	4.7%	-0.001	0.043	3.8%	0.000	0.043	4.0%
	$\lambda_0 = 0.4$	-0.091	0.370	15.9%	0.091	0.291	16.4%	-0.001	0.241	13.5%
	$\rho_0 = 0.3$	-0.017	0.392	18.8%	-0.134	0.442	14.8%	-0.189	0.374	10.1%
	$\beta_{10} = 0.8$	2.054	106.014	12.4%	-0.211	0.698	14.0%	0.003	0.590	11.5%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.002	0.043	4.1%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.087	0.348	13.9%	0.039	0.244	10.0%	-0.020	0.231	11.6%
	$\rho_0 = 0.1$	-0.003	0.421	16.0%	-0.067	0.420	10.0%	-0.174	0.387	9.9%
	$\beta_{10} = 0.8$	1.372	89.569	11.0%	-0.089	0.592	8.4%	0.048	0.567	9.8%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.1%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.085	0.336	12.9%	0.021	0.228	7.8%	-0.028	0.224	10.2%
	$\rho_0 = 0$	0.003	0.432	14.8%	-0.038	0.414	8.5%	-0.161	0.387	9.4%
	$\beta_{10} = 0.8$	0.685	83.668	10.3%	-0.047	0.555	6.6%	0.067	0.551	8.4%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.6%	-0.001	0.042	4.0%	-0.001	0.043	4.4%

Table 2.A9: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 = -0.4$, $\rho_0 \geq 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 10$	$\lambda_0 = -0.4$	-0.002	0.116	5.5%	0.366	0.519	19.8%	0.031	0.144	7.3%
	$\rho_0 = 0.9$	-0.001	0.022	7.0%	-0.046	0.177	17.1%	-0.011	0.038	4.2%
	$\beta_{10} = 0.8$	0.001	0.532	4.8%	-0.377	0.698	13.8%	-0.035	0.530	6.4%
	$\beta_{20} = 0.2$	-0.001	0.049	5.6%	0.002	0.073	3.2%	0.000	0.049	5.3%
	$\beta_{30} = 1.5$	-0.002	0.048	4.7%	0.014	0.076	4.6%	0.006	0.050	5.5%
	$\lambda_0 = -0.4$	-0.003	0.127	6.0%	0.060	0.222	9.4%	0.022	0.137	6.7%
	$\rho_0 = 0.7$	-0.003	0.054	6.6%	-0.034	0.078	6.5%	-0.019	0.064	5.1%
	$\beta_{10} = 0.8$	0.004	0.262	5.2%	-0.062	0.335	6.7%	-0.024	0.265	5.7%
	$\beta_{20} = 0.2$	-0.001	0.050	5.6%	0.000	0.059	5.2%	0.000	0.050	5.4%
	$\beta_{30} = 1.5$	-0.002	0.048	4.9%	-0.001	0.059	4.6%	0.003	0.048	4.7%
	$\lambda_0 = -0.4$	-0.005	0.122	6.4%	0.010	0.131	6.4%	0.011	0.121	6.7%
	$\rho_0 = 0.3$	-0.005	0.100	7.1%	-0.030	0.102	5.4%	-0.023	0.103	6.2%
	$\beta_{10} = 0.8$	0.006	0.210	5.7%	-0.011	0.217	5.4%	-0.010	0.209	5.5%
	$\beta_{20} = 0.2$	-0.001	0.051	5.7%	-0.000	0.051	5.4%	-0.000	0.051	5.5%
	$\beta_{30} = 1.5$	-0.002	0.045	5.0%	-0.001	0.046	4.8%	0.000	0.045	4.7%
	$\lambda_0 = -0.4$	-0.005	0.113	6.2%	0.005	0.114	5.7%	0.008	0.111	6.4%
	$\rho_0 = 0.1$	-0.005	0.113	6.8%	-0.023	0.110	4.9%	-0.023	0.113	6.1%
	$\beta_{10} = 0.8$	0.006	0.200	5.7%	-0.005	0.199	5.2%	-0.007	0.197	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.002	0.044	5.0%	-0.001	0.044	4.7%	-0.001	0.044	4.7%
	$\lambda_0 = -0.4$	-0.005	0.109	6.1%	0.004	0.108	5.5%	0.006	0.101	5.4%
	$\rho_0 = 0$	-0.005	0.117	6.6%	-0.020	0.114	5.0%	-0.020	0.113	5.8%
	$\beta_{10} = 0.8$	0.005	0.195	5.6%	-0.003	0.193	5.1%	-0.006	0.189	4.9%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.6%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.002	0.044	5.1%	-0.001	0.043	4.8%	-0.001	0.043	4.8%
$J = 20$	$\lambda_0 = -0.4$	-0.001	0.180	5.7%	0.615	0.765	39.3%	0.063	0.217	4.3%
	$\rho_0 = 0.9$	-0.004	0.035	8.7%	-0.186	0.374	23.7%	-0.023	0.061	2.8%
	$\beta_{10} = 0.8$	0.024	2.667	5.4%	-0.624	0.858	34.4%	-0.067	0.547	7.3%
	$\beta_{20} = 0.2$	-0.000	0.050	5.5%	0.002	0.058	3.6%	0.001	0.050	5.3%
	$\beta_{30} = 1.5$	-0.001	0.046	5.1%	0.017	0.057	5.3%	0.006	0.047	4.7%
	$\lambda_0 = -0.4$	-0.003	0.196	7.0%	0.137	0.329	15.5%	0.050	0.208	6.9%
	$\rho_0 = 0.7$	-0.009	0.083	8.5%	-0.075	0.137	7.3%	-0.040	0.099	4.7%
	$\beta_{10} = 0.8$	0.003	0.302	5.5%	-0.139	0.403	11.1%	-0.052	0.308	6.0%
	$\beta_{20} = 0.2$	-0.000	0.050	5.6%	0.000	0.054	5.0%	0.000	0.050	5.3%
	$\beta_{30} = 1.5$	-0.002	0.046	5.2%	0.000	0.050	4.3%	0.004	0.046	4.8%
	$\lambda_0 = -0.4$	-0.010	0.184	7.9%	0.025	0.194	7.9%	0.023	0.179	8.5%
	$\rho_0 = 0.3$	-0.012	0.145	8.5%	-0.058	0.150	5.3%	-0.048	0.151	7.0%
	$\beta_{10} = 0.8$	0.010	0.252	6.1%	-0.025	0.260	5.9%	-0.023	0.247	6.2%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.4%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.002	0.045	5.2%	-0.001	0.045	4.7%	0.000	0.044	4.8%
	$\lambda_0 = -0.4$	-0.011	0.171	7.4%	0.013	0.168	6.6%	0.015	0.163	7.9%
	$\rho_0 = 0.1$	-0.011	0.163	7.9%	-0.046	0.158	4.9%	-0.047	0.165	7.0%
	$\beta_{10} = 0.8$	0.011	0.237	5.9%	-0.013	0.235	5.4%	-0.015	0.231	5.7%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.6%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.002	0.044	5.0%	-0.001	0.044	4.8%	-0.001	0.043	4.7%
	$\lambda_0 = -0.4$	-0.011	0.163	7.3%	0.010	0.159	6.2%	0.010	0.150	6.8%
	$\rho_0 = 0$	-0.011	0.169	7.4%	-0.040	0.163	4.8%	-0.041	0.164	6.4%
	$\beta_{10} = 0.8$	0.011	0.230	5.8%	-0.010	0.226	5.3%	-0.009	0.219	5.2%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.002	0.044	5.0%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
$J = 100$	$\lambda_0 = -0.4$	0.031	0.423	5.6%	1.098	1.197	75.7%	0.333	0.540	13.4%
	$\rho_0 = 0.9$	-0.037	0.172	24.8%	-0.701	0.842	44.6%	-0.169	0.267	5.0%
	$\beta_{10} = 0.8$	-0.271	95.603	9.2%	-1.102	1.225	73.7%	-0.340	0.690	16.5%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	0.001	0.051	5.1%	0.001	0.051	4.9%
	$\beta_{30} = 1.5$	-0.001	0.044	4.8%	0.009	0.045	5.1%	0.005	0.044	4.2%
	$\lambda_0 = -0.4$	0.009	0.429	8.6%	0.586	0.806	43.6%	0.250	0.466	18.2%
	$\rho_0 = 0.7$	-0.050	0.274	21.0%	-0.440	0.597	20.7%	-0.242	0.372	10.2%
	$\beta_{10} = 0.8$	-1.977	116.313	8.3%	-0.588	0.829	39.8%	-0.252	0.511	15.7%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	0.001	0.051	5.3%	0.000	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.044	4.8%	0.004	0.044	4.5%	0.003	0.043	4.3%
	$\lambda_0 = -0.4$	-0.023	0.394	9.3%	0.175	0.460	16.4%	0.111	0.358	16.0%
	$\rho_0 = 0.3$	-0.035	0.383	18.5%	-0.222	0.448	11.1%	-0.252	0.428	12.6%
	$\beta_{10} = 0.8$	-0.684	99.172	7.7%	-0.175	0.487	13.7%	-0.111	0.395	12.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.4%	-0.000	0.051	5.5%
	$\beta_{30} = 1.5$	-0.001	0.043	4.9%	0.000	0.043	4.5%	0.000	0.043	4.6%
	$\lambda_0 = -0.4$	-0.032	0.372	8.9%	0.086	0.390	10.4%	0.067	0.325	13.7%
	$\rho_0 = 0.1$	-0.024	0.415	17.1%	-0.143	0.412	7.7%	-0.236	0.431	11.8%
	$\beta_{10} = 0.8$	0.053	81.565	6.9%	-0.086	0.420	7.9%	-0.067	0.363	10.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.000	0.043	4.6%	-0.000	0.043	4.6%
	$\lambda_0 = -0.4$	-0.035	0.362	8.5%	0.054	0.368	7.9%	0.047	0.307	11.7%
	$\rho_0 = 0$	-0.018	0.426	16.3%	-0.106	0.401	6.5%	-0.222	0.424	10.9%
	$\beta_{10} = 0.8$	-0.325	74.845	6.7%	-0.054	0.400	6.2%	-0.047	0.344	8.6%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.6%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.000	0.043	4.5%	-0.001	0.043	4.6%

Table 2.A10: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 = 0.4$, $\rho_0 < 0$)

		GMM			GS2SLS			II		
	$\boldsymbol{\theta}_0$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$	Bias	RMSE	$P(5\%)$
$J = 10$	$\lambda_0 = 0.4$	-0.004	0.039	3.9%	-0.009	0.046	3.6%	-0.070	0.239	3.5%
	$\rho_0 = -0.9$	0.007	0.096	2.1%	0.040	0.101	3.0%	0.103	0.362	2.3%
	$\beta_{10} = 0.8$	0.011	0.172	5.3%	0.020	0.195	4.9%	0.165	0.591	5.1%
	$\beta_{20} = 0.2$	-0.001	0.049	5.4%	0.000	0.055	5.2%	-0.001	0.050	5.0%
	$\beta_{30} = 1.5$	0.000	0.046	4.7%	0.003	0.047	4.4%	-0.001	0.054	4.6%
	$\lambda_0 = 0.4$	-0.002	0.044	5.1%	-0.001	0.048	4.6%	-0.014	0.100	4.7%
	$\rho_0 = -0.7$	-0.006	0.117	5.3%	0.004	0.115	4.3%	0.010	0.180	5.7%
	$\beta_{10} = 0.8$	0.007	0.180	5.7%	0.004	0.194	5.1%	0.035	0.278	5.3%
	$\beta_{20} = 0.2$	-0.001	0.050	5.4%	-0.001	0.053	5.1%	-0.001	0.050	5.3%
	$\beta_{30} = 1.5$	-0.001	0.045	5.0%	-0.001	0.046	4.6%	-0.001	0.046	4.8%
	$\lambda_0 = 0.4$	-0.003	0.054	5.3%	0.001	0.055	4.9%	-0.001	0.055	5.4%
	$\rho_0 = -0.3$	-0.005	0.117	5.2%	-0.004	0.116	4.9%	-0.015	0.119	5.4%
	$\beta_{10} = 0.8$	0.010	0.199	5.3%	0.001	0.202	5.0%	0.005	0.200	5.2%
	$\beta_{20} = 0.2$	-0.001	0.051	5.5%	-0.001	0.051	5.1%	-0.001	0.050	5.3%
	$\beta_{30} = 1.5$	-0.001	0.044	4.9%	-0.001	0.044	4.7%	-0.001	0.044	4.7%
	$\lambda_0 = 0.4$	-0.004	0.057	5.2%	0.001	0.058	4.9%	-0.001	0.057	5.5%
	$\rho_0 = -0.2$	-0.005	0.117	5.2%	-0.005	0.115	4.7%	-0.016	0.117	5.4%
	$\beta_{10} = 0.8$	0.011	0.205	5.2%	0.001	0.206	5.0%	0.004	0.204	5.2%
	$\beta_{20} = 0.2$	-0.001	0.051	5.5%	-0.001	0.051	5.1%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.001	0.044	4.8%	-0.001	0.044	4.7%	-0.001	0.043	4.7%
	$\lambda_0 = 0.4$	-0.004	0.061	5.2%	0.001	0.061	5.1%	-0.001	0.060	5.4%
	$\rho_0 = -0.1$	-0.004	0.116	5.3%	-0.007	0.114	4.8%	-0.016	0.116	5.4%
	$\beta_{10} = 0.8$	0.013	0.213	5.2%	-0.000	0.212	4.9%	0.004	0.210	5.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.5%	-0.001	0.051	5.1%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.6%	-0.001	0.043	4.6%
$J = 20$	$\lambda_0 = 0.4$	-0.009	0.053	3.1%	-0.014	0.060	3.3%	-0.062	0.207	3.2%
	$\rho_0 = -0.9$	0.026	0.134	2.3%	0.079	0.149	3.5%	0.086	0.323	2.0%
	$\beta_{10} = 0.8$	0.023	0.194	4.7%	0.036	0.211	4.3%	0.145	0.510	4.4%
	$\beta_{20} = 0.2$	-0.001	0.050	5.3%	-0.001	0.052	5.0%	-0.000	0.050	5.0%
	$\beta_{30} = 1.5$	0.001	0.044	4.5%	0.003	0.045	4.5%	0.003	0.046	4.5%
	$\lambda_0 = 0.4$	-0.006	0.059	4.7%	-0.003	0.063	3.9%	-0.018	0.117	4.3%
	$\rho_0 = -0.7$	-0.003	0.166	2.4%	0.012	0.159	2.5%	-0.001	0.217	2.1%
	$\beta_{10} = 0.8$	0.016	0.206	5.1%	0.010	0.215	4.6%	0.043	0.313	4.6%
	$\beta_{20} = 0.2$	-0.001	0.050	5.5%	-0.001	0.052	5.0%	-0.001	0.050	5.3%
	$\beta_{30} = 1.5$	-0.001	0.044	4.5%	-0.000	0.044	4.4%	-0.001	0.044	4.4%
	$\lambda_0 = 0.4$	-0.009	0.076	5.1%	0.001	0.076	5.3%	-0.003	0.075	5.8%
	$\rho_0 = -0.3$	-0.006	0.169	5.3%	-0.011	0.167	4.8%	-0.030	0.170	5.7%
	$\beta_{10} = 0.8$	0.022	0.237	5.1%	0.000	0.237	4.8%	0.010	0.235	5.0%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.2%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	-0.001	0.043	4.4%	-0.001	0.043	4.5%
	$\lambda_0 = 0.4$	-0.010	0.082	5.3%	0.002	0.080	5.5%	-0.003	0.078	6.0%
	$\rho_0 = -0.2$	-0.005	0.168	5.4%	-0.014	0.166	4.7%	-0.032	0.168	5.6%
	$\beta_{10} = 0.8$	0.025	0.249	5.1%	-0.002	0.245	4.8%	0.009	0.242	5.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	-0.001	0.043	4.4%	-0.001	0.043	4.5%
	$\lambda_0 = 0.4$	-0.011	0.089	5.4%	0.002	0.085	5.6%	-0.002	0.083	6.4%
	$\rho_0 = -0.1$	-0.005	0.166	5.6%	-0.016	0.164	4.7%	-0.034	0.165	5.8%
	$\beta_{10} = 0.8$	0.029	0.264	5.0%	-0.003	0.255	4.8%	0.007	0.251	5.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.4%	-0.001	0.051	5.1%	-0.001	0.051	5.1%
	$\beta_{30} = 1.5$	-0.001	0.043	4.5%	-0.001	0.043	4.4%	-0.001	0.043	4.4%
$J = 100$	$\lambda_0 = 0.4$	-0.052	0.192	3.9%	-0.018	0.163	3.5%	-0.175	0.372	2.9%
	$\rho_0 = -0.9$	0.142	0.356	4.7%	0.196	0.375	5.6%	0.189	0.493	2.3%
	$\beta_{10} = 0.8$	0.136	31.221	3.9%	0.043	0.409	3.6%	0.410	0.883	3.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.2%	-0.000	0.051	4.8%
	$\beta_{30} = 1.5$	-0.000	0.043	4.6%	0.000	0.043	4.6%	0.001	0.043	4.1%
	$\lambda_0 = 0.4$	-0.054	0.224	5.4%	-0.011	0.171	4.2%	-0.119	0.304	3.1%
	$\rho_0 = -0.7$	0.078	0.394	5.6%	0.110	0.376	5.2%	0.042	0.415	2.0%
	$\beta_{10} = 0.8$	-0.287	42.068	5.2%	0.027	0.425	4.1%	0.278	0.729	3.4%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.1%	-0.001	0.051	5.0%
	$\beta_{30} = 1.5$	-0.001	0.043	4.7%	-0.001	0.043	4.3%	0.000	0.043	4.2%
	$\lambda_0 = 0.4$	-0.069	0.289	9.7%	0.008	0.196	6.4%	-0.046	0.227	6.9%
	$\rho_0 = -0.3$	0.019	0.444	8.1%	-0.010	0.412	5.1%	-0.131	0.397	3.6%
	$\beta_{10} = 0.8$	1.143	65.945	8.1%	-0.016	0.484	5.7%	0.109	0.555	6.2%
	$\beta_{20} = 0.2$	-0.001	0.051	5.3%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.2%	-0.001	0.043	4.3%
	$\lambda_0 = 0.4$	-0.075	0.305	10.7%	0.015	0.206	7.5%	-0.038	0.225	8.3%
	$\rho_0 = -0.2$	0.013	0.444	10.3%	-0.032	0.420	5.7%	-0.148	0.400	6.5%
	$\beta_{10} = 0.8$	0.933	69.677	8.8%	-0.033	0.506	6.4%	0.090	0.551	7.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.3%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.2%	-0.001	0.043	4.4%
	$\lambda_0 = 0.4$	-0.079	0.318	11.8%	0.023	0.217	8.8%	-0.031	0.225	9.8%
	$\rho_0 = -0.1$	0.007	0.439	13.1%	-0.051	0.429	7.8%	-0.159	0.396	8.4%
	$\beta_{10} = 0.8$	0.349	76.675	9.6%	-0.053	0.531	7.3%	0.075	0.550	8.1%
	$\beta_{20} = 0.2$	-0.001	0.051	5.2%	-0.001	0.051	5.0%	-0.001	0.051	5.2%
	$\beta_{30} = 1.5$	-0.002	0.043	4.7%	-0.001	0.043	4.2%	-0.001	0.043	4.3%

Table 2.A11: GMM, GS2SLS, and II under Circular Weight Matrices ($n = 1000$, $\lambda_0 < 0$, $\rho_0 = -0.4$)

	$\boldsymbol{\theta}_0$	GMM Bias	GMM RMSE	GMM $P(5\%)$	GS2SLS Bias	GS2SLS RMSE	GS2SLS $P(5\%)$	II Bias	II RMSE	II $P(5\%)$
$J = 10$	$\lambda_0 = -0.9$	0.007	0.089	3.3%	0.000	0.110	5.0%	0.015	0.090	3.7%
	$\rho_0 = -0.4$	-0.014	0.120	4.3%	-0.013	0.127	4.9%	-0.029	0.122	4.6%
	$\beta_{10} = 0.8$	-0.006	0.168	4.8%	0.001	0.178	5.2%	-0.011	0.168	4.8%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.050	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	5.0%	-0.001	0.043	4.8%	-0.001	0.043	4.8%
	$\lambda_0 = -0.7$	-0.003	0.101	5.9%	0.000	0.105	4.9%	0.005	0.100	5.8%
	$\rho_0 = -0.4$	-0.006	0.129	6.3%	-0.011	0.127	4.9%	-0.020	0.128	6.0%
	$\beta_{10} = 0.8$	0.002	0.176	5.4%	0.001	0.180	5.2%	-0.004	0.174	5.2%
	$\beta_{20} = 0.2$	-0.000	0.051	5.7%	-0.000	0.051	5.5%	-0.000	0.050	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	5.0%	-0.001	0.043	4.8%	-0.001	0.043	4.8%
	$\lambda_0 = -0.3$	-0.004	0.089	5.7%	0.000	0.092	5.0%	0.003	0.088	5.8%
	$\rho_0 = -0.4$	-0.006	0.126	6.0%	-0.007	0.125	5.1%	-0.019	0.126	6.0%
	$\beta_{10} = 0.8$	0.004	0.181	5.3%	0.001	0.185	5.3%	-0.003	0.180	5.1%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	-0.000	0.051	5.4%	-0.000	0.050	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.9%	-0.001	0.043	4.8%	-0.001	0.043	4.6%
	$\lambda_0 = -0.2$	-0.004	0.085	5.7%	-0.000	0.087	5.1%	0.002	0.084	5.7%
	$\rho_0 = -0.4$	-0.006	0.126	6.0%	-0.006	0.124	5.1%	-0.019	0.125	5.9%
	$\beta_{10} = 0.8$	0.004	0.182	5.3%	0.001	0.186	5.2%	-0.002	0.181	5.1%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	-0.000	0.052	5.4%	-0.000	0.050	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.7%	-0.001	0.043	4.6%
	$\lambda_0 = -0.1$	-0.004	0.080	5.6%	0.000	0.083	5.3%	0.001	0.079	5.7%
	$\rho_0 = -0.4$	-0.006	0.124	5.9%	-0.005	0.123	5.3%	-0.019	0.124	5.8%
	$\beta_{10} = 0.8$	0.005	0.184	5.3%	0.001	0.188	5.2%	-0.002	0.182	5.1%
	$\beta_{20} = 0.2$	0.000	0.051	5.7%	-0.000	0.052	5.5%	-0.000	0.050	5.6%
	$\beta_{30} = 1.5$	-0.001	0.044	4.8%	-0.001	0.044	4.7%	-0.001	0.044	4.6%
$J = 20$	$\lambda_0 = -0.9$	0.022	0.119	3.9%	0.003	0.162	5.0%	0.033	0.123	4.7%
	$\rho_0 = -0.4$	-0.031	0.170	4.0%	-0.027	0.179	4.5%	-0.063	0.175	4.8%
	$\beta_{10} = 0.8$	-0.016	0.178	4.2%	-0.002	0.198	5.0%	-0.024	0.179	4.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.5%
	$\beta_{30} = 1.5$	-0.000	0.043	4.9%	-0.001	0.043	4.8%	-0.001	0.043	4.7%
	$\lambda_0 = -0.7$	-0.005	0.144	4.3%	0.003	0.152	5.0%	0.010	0.141	6.0%
	$\rho_0 = -0.4$	-0.012	0.183	6.0%	-0.024	0.179	4.7%	-0.044	0.184	6.0%
	$\beta_{10} = 0.8$	0.004	0.195	5.2%	-0.002	0.201	5.0%	-0.008	0.194	5.1%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.9%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
	$\lambda_0 = -0.3$	-0.009	0.129	6.0%	0.002	0.129	5.2%	0.003	0.124	6.2%
	$\rho_0 = -0.4$	-0.010	0.184	6.3%	-0.018	0.178	4.7%	-0.040	0.183	6.2%
	$\beta_{10} = 0.8$	0.010	0.208	5.6%	-0.002	0.209	5.0%	-0.003	0.204	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.6%	-0.001	0.043	4.6%
	$\lambda_0 = -0.2$	-0.009	0.122	5.9%	0.002	0.122	5.1%	0.002	0.117	6.1%
	$\rho_0 = -0.4$	-0.010	0.183	6.1%	-0.017	0.177	4.8%	-0.040	0.181	6.1%
	$\beta_{10} = 0.8$	0.011	0.210	5.4%	-0.002	0.211	5.0%	-0.002	0.206	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.9%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
	$\lambda_0 = -0.1$	-0.009	0.115	5.7%	0.002	0.114	5.1%	0.001	0.110	6.1%
	$\rho_0 = -0.4$	-0.010	0.182	6.0%	-0.016	0.177	4.9%	-0.039	0.180	6.1%
	$\beta_{10} = 0.8$	0.012	0.212	5.4%	-0.002	0.213	5.0%	-0.001	0.208	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	-0.001	0.043	4.9%	-0.001	0.043	4.7%	-0.001	0.043	4.7%
$J = 100$	$\lambda_0 = -0.9$	0.081	0.231	4.6%	0.015	0.393	5.6%	0.093	0.223	4.4%
	$\rho_0 = -0.4$	-0.047	0.376	3.7%	-0.031	0.393	5.2%	-0.234	0.377	2.1%
	$\beta_{10} = 0.8$	-0.324	27.040	3.6%	-0.010	0.328	4.6%	-0.069	0.225	3.5%
	$\beta_{20} = 0.2$	0.000	0.051	5.8%	-0.000	0.051	5.5%	-0.000	0.051	5.7%
	$\beta_{30} = 1.5$	0.000	0.043	4.9%	-0.000	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.7$	0.011	0.268	5.0%	0.014	0.368	5.6%	0.034	0.243	4.3%
	$\rho_0 = -0.4$	-0.009	0.398	6.0%	-0.018	0.399	5.5%	-0.196	0.374	2.1%
	$\beta_{10} = 0.8$	-0.135	33.490	3.8%	-0.011	0.340	4.7%	-0.028	0.253	4.0%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.000	0.043	4.8%	-0.000	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.3$	-0.052	0.321	5.7%	0.014	0.311	5.9%	-0.013	0.260	6.5%
	$\rho_0 = -0.4$	0.024	0.437	10.1%	-0.003	0.405	5.7%	-0.165	0.380	2.5%
	$\beta_{10} = 0.8$	-0.708	46.302	4.8%	-0.015	0.367	4.8%	0.014	0.319	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.2$	-0.059	0.323	6.3%	0.013	0.293	5.9%	-0.021	0.258	6.5%
	$\rho_0 = -0.4$	0.026	0.441	10.2%	0.002	0.407	5.6%	-0.159	0.381	2.6%
	$\beta_{10} = 0.8$	-0.574	51.389	5.1%	-0.014	0.374	4.8%	0.024	0.338	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.6%	-0.000	0.043	4.7%
	$\lambda_0 = -0.1$	-0.063	0.323	7.8%	0.012	0.277	5.8%	-0.026	0.252	6.3%
	$\rho_0 = -0.4$	0.028	0.445	10.1%	0.004	0.407	5.5%	-0.153	0.382	2.6%
	$\beta_{10} = 0.8$	-0.337	54.842	6.0%	-0.015	0.384	4.9%	0.034	0.356	5.3%
	$\beta_{20} = 0.2$	-0.000	0.051	5.9%	-0.000	0.051	5.5%	-0.000	0.051	5.6%
	$\beta_{30} = 1.5$	-0.001	0.043	4.8%	-0.001	0.043	4.5%	-0.000	0.043	4.7%

Table 2.A12: GMM, GS2SLS, and II under Circular Weight Matrices
($n = 5000,\ J = 100,\ \lambda_0 = \pm 0.4,\ \rho_0 \geq 0$)

	GMM			GS2SLS			II		
$\boldsymbol{\theta}_0$	Bias	RMSE	P(5%)	Bias	RMSE	P(5%)	Bias	RMSE	P(5%)
$\lambda_0 = 0.4$	0.001	0.182	9.2%	0.349	0.439	54.7%	0.107	0.206	11.7%
$\rho_0 = 0.9$	-0.019	0.080	13.7%	-0.226	0.321	34.7%	-0.064	0.112	4.5%
$\beta_{10} = 0.8$	-0.007	0.502	8.2%	-0.818	1.034	53.6%	-0.253	0.526	11.5%
$\beta_{20} = 0.2$	-0.001	0.022	4.4%	-0.001	0.022	4.6%	-0.001	0.022	3.5%
$\beta_{30} = 1.5$	0.000	0.019	5.1%	-0.001	0.019	4.3%	0.002	0.019	4.8%
$\lambda_0 = 0.4$	-0.013	0.177	10.7%	0.076	0.210	18.6%	0.044	0.163	15.5%
$\rho_0 = 0.7$	-0.021	0.124	11.9%	-0.093	0.173	11.1%	-0.064	0.137	9.2%
$\beta_{10} = 0.8$	0.032	0.429	10.6%	-0.175	0.499	18.3%	-0.102	0.395	14.7%
$\beta_{20} = 0.2$	-0.001	0.022	4.4%	-0.001	0.022	5.0%	-0.001	0.022	4.2%
$\beta_{30} = 1.5$	-0.000	0.019	5.1%	-0.001	0.019	5.2%	0.001	0.019	5.5%
$\lambda_0 = 0.4$	-0.015	0.134	8.1%	0.015	0.117	9.0%	0.009	0.115	8.1%
$\rho_0 = 0.3$	-0.013	0.158	7.9%	-0.038	0.160	6.3%	-0.048	0.157	6.5%
$\beta_{10} = 0.8$	0.037	0.326	7.8%	-0.033	0.283	7.5%	-0.018	0.281	8.1%
$\beta_{20} = 0.2$	-0.001	0.022	4.3%	-0.001	0.022	4.7%	-0.001	0.022	4.6%
$\beta_{30} = 1.5$	-0.000	0.019	5.5%	-0.000	0.019	5.5%	0.000	0.019	5.6%
$\lambda_0 = 0.4$	-0.011	0.111	7.0%	0.010	0.095	7.1%	0.002	0.094	7.2%
$\rho_0 = 0.1$	-0.011	0.167	6.7%	-0.031	0.164	5.3%	-0.040	0.160	6.4%
$\beta_{10} = 0.8$	0.027	0.271	6.7%	-0.021	0.234	6.1%	-0.003	0.231	6.7%
$\beta_{20} = 0.2$	-0.001	0.022	4.4%	-0.001	0.022	4.8%	-0.001	0.022	4.6%
$\beta_{30} = 1.5$	0.000	0.019	5.6%	0.000	0.019	5.7%	0.000	0.019	5.6%
$\lambda_0 = 0.4$	-0.008	0.100	6.5%	0.009	0.088	6.8%	0.001	0.084	6.4%
$\rho_0 = 0$	-0.012	0.170	6.0%	-0.029	0.167	5.1%	-0.035	0.159	6.5%
$\beta_{10} = 0.8$	0.022	0.246	6.0%	-0.018	0.217	5.9%	0.001	0.209	6.2%
$\beta_{20} = 0.2$	-0.001	0.022	4.6%	-0.001	0.022	4.5%	-0.001	0.022	4.7%
$\beta_{30} = 1.5$	0.000	0.019	5.4%	0.000	0.019	5.5%	0.000	0.019	5.6%
$\lambda_0 = -0.4$	0.010	0.193	5.3%	0.635	0.799	39.9%	0.075	0.226	4.3%
$\rho_0 = 0.9$	-0.005	0.034	6.7%	-0.196	0.396	26.1%	-0.023	0.057	3.5%
$\beta_{10} = 0.8$	-0.008	0.298	5.5%	-0.633	0.816	39.2%	-0.073	0.315	6.7%
$\beta_{20} = 0.2$	-0.001	0.022	3.9%	-0.001	0.022	3.4%	-0.001	0.022	3.9%
$\beta_{30} = 1.5$	-0.001	0.019	4.6%	0.003	0.020	4.2%	0.001	0.020	4.8%
$\lambda_0 = -0.4$	0.007	0.208	6.6%	0.151	0.345	15.3%	0.064	0.223	7.9%
$\rho_0 = 0.7$	-0.011	0.081	6.7%	-0.075	0.144	6.8%	-0.042	0.098	4.7%
$\beta_{10} = 0.8$	-0.004	0.229	5.9%	-0.148	0.356	13.6%	-0.062	0.241	5.7%
$\beta_{20} = 0.2$	-0.001	0.022	4.0%	-0.001	0.022	4.6%	-0.001	0.022	4.0%
$\beta_{30} = 1.5$	-0.001	0.020	5.0%	-0.001	0.020	4.8%	0.000	0.020	4.8%
$\lambda_0 = -0.4$	-0.004	0.191	7.2%	0.029	0.204	8.1%	0.029	0.187	8.7%
$\rho_0 = 0.3$	-0.012	0.138	5.8%	-0.055	0.145	4.3%	-0.047	0.147	6.4%
$\beta_{10} = 0.8$	0.007	0.205	6.4%	-0.026	0.216	6.8%	-0.026	0.200	7.1%
$\beta_{20} = 0.2$	-0.001	0.022	4.0%	-0.001	0.022	4.7%	-0.001	0.022	4.1%
$\beta_{30} = 1.5$	-0.001	0.019	5.1%	-0.001	0.019	5.0%	-0.001	0.019	5.1%
$\lambda_0 = -0.4$	-0.007	0.175	7.6%	0.017	0.175	6.7%	0.018	0.167	7.9%
$\rho_0 = 0.1$	-0.009	0.155	5.6%	-0.043	0.153	4.0%	-0.044	0.159	5.8%
$\beta_{10} = 0.8$	0.011	0.188	6.2%	-0.014	0.188	5.5%	-0.015	0.181	6.6%
$\beta_{20} = 0.2$	-0.001	0.022	3.8%	-0.001	0.022	4.2%	-0.001	0.022	4.1%
$\beta_{30} = 1.5$	-0.001	0.019	5.1%	-0.001	0.019	5.1%	-0.001	0.019	5.0%
$\lambda_0 = -0.4$	-0.008	0.167	7.0%	0.013	0.166	6.1%	0.014	0.156	7.6%
$\rho_0 = 0$	-0.008	0.162	5.3%	-0.037	0.159	3.9%	-0.041	0.162	6.0%
$\beta_{10} = 0.8$	0.011	0.180	6.0%	-0.010	0.179	5.5%	-0.011	0.169	6.5%
$\beta_{20} = 0.2$	-0.001	0.022	3.8%	-0.001	0.022	4.2%	-0.001	0.022	4.2%
$\beta_{30} = 1.5$	-0.001	0.019	5.1%	-0.001	0.019	5.1%	-0.001	0.019	4.9%

Appendix C: Matlab Code

The following is the Matlab code (with line number added) used in this paper to implement the indirect inference estimation of SARA(1,1).

```
% Matlab source code to implement II estimation of SARAR(1,1)
% Bao and Liu, Spatial Economic Analysis, "Estimating a Spatial Autoregressive Model with ...
%   Autoregressive Disturbances Based on the Indirect Inference Principle"
%
% INPUTS
%   y, X:       sample data y X
%   W, M:       spatial weights matrices W and M
%   wm:         matrix norms of W and M
%   lamrho0:    initial SARA(1,1) parameters consisting of [lam;rho]
%   options0:   options for fmincon to search for parameters
% OUTPUTS
%   thetaII:    II estimator of [beta;lam;rho]
%   se:         standard error of thetaII
%   inegative:  indicator for negative estimated variance matrix
%
% REMARKS
%   This code uses "fmincon" to minimize (quadratic) distance based on the sample...
%       binding function of the II estimator
%   With row-normalized weight matrices, lam and rho are bounded by 1 (wm=[1,1]) in magnitude;
%   Parameter constraints by fminconstr;
%
function [thetaII,se,inegative]=II_SARAR(y,X,W,M,wm,lamrho0,options0)
    inegative=0; % to record whether components of estimated variance are negative or not
    [n,k]=size(X);
    thetaII=NaN*zeros(k+2,1);
    se=thetaII;
    if isempty(options0) % options for fminsearch with gradient specified
        options0=optimoptions('fmincon','TolX',1e-20,'MaxFunEvals',5000,'Display','off',...
            'SpecifyObjectiveGradient',true);
    end
    if isempty(lamrho0)
        lamrho0=[0.1; 0.1]; % as column vector, initial values for lambda and rho
    end
    warning('off','all');
    [lamrhoII,¬,exitflag]=fmincon(@(lamrho)IIsolve(lamrho,y,X,W,M),lamrho0,...
        [],[],[],[],[],[],@(lamrho)fminconstrII(lamrho,wm),options0);
    if exitflag>0 % fsolve successful
        I=speye(n); % sparse identity matrix
        S=I-W*lamrhoII(1);
        R=I-M*lamrhoII(2);
        RX=R*X;
        betaII=regress(R*S*y,RX);
        thetaII=[betaII;lamrhoII]; % return as a column vector
        V=IIV(thetaII,y,X,W,M); % estimated variance matrix of II estimator
        se=sqrt(diag(V));
    end
    if ¬isreal(se)
        se=zeros(k+2,1);
        inegative=1;
    end
end

% To be used by "fsolve" or "fmincon" for the II procedure
% The second output is the gradient of quadratic objection function
% The third output is the gradient of sample binding function
% B,D,F,G,H,K,R,GR,XRRXinvXR,v are to be used in calculating the variance of the II estimator
function [f2,g,B,D,F,G,H,K,R,GR,XRRXinvXR,v]=IIsolve(lamrho,y,X,W,M)
    n=length(y);
    I=speye(n);
    R=I-M*lamrho(2);
```

```
    dR=decomposition(R,'lu');
    F=M/dR;
    K=diag(F);
    RX=R*X;
    XRRXinvXR=(RX/(RX'*RX))'; % (X'R'RX)^{-1}*(X'R')
    H=I-RX*XRRXinvXR;
    HR=H*R;

    S=I-W*lamrho(1);
    dS=decomposition(S,'lu');
    G=W/dS;
    GR=G/dR;
    Ry=R*y;
    D=dg(HR,GR);
    Wy=W*y;
    HRWy=HR*Wy;

    RS=R*S;
    RSy=RS*y;
    [¬,¬,v]=regress(RSy,RX); % possibly faster than H*R*S*y

    Fv=F*v;
    RRv=dR\v;
    % sample binding function related to lambda
    flam=(sum(HRWy.*Ry)-sum(D.*(v.^2)))/sum(HRWy.^2)-lamrho(1);
    % sample binding function related to rho
    frho=(sum(RRv.*Fv)-sum(K.*(v.^2)))./sum(Fv.^2,1)-lamrho(2);
    f2=flam^2+frho^2;

    if nargout > 1 % gradient w.r.t. lambda and rho
        Blam=(2*sum(D.*HRWy.*v)-sum(dg(HR,G*GR).*(v.^2)))/sum(HRWy.^2)-1;
        Sy=S*y;
        Hrho=H*M*X*XRRXinvXR;
        Hrho=Hrho+Hrho'; % dirative of H w.r.t. rho_{j}
        HM=H*M;
        t1=Hrho*R-HM;
        tempj=(F')/dR;
        FF=F*F;
        t2=tempj*F+FF'/dR;
        t3=t1'*tempj+2*(HM.*K)';
        t4=2*Hrho.*K+H.*diag(FF);
        ej1j2=2*sum((F*Hrho*RSy-F*HM*Sy+FF*v).*Fv);
        ej1=sum(Fv.^2);
        eji=-2*sum(Fv.*(F*HRWy));
        RWy=R*Wy;
        dij=sum(RWy.*(Hrho*RWy))-2*sum((M*Wy).*HRWy);
        Dij=dg(Hrho*R-H*M,GR)+dg(HR,GR*F);
        di=sum(HRWy.^2);
        Brho=(sum(((dR\t1)*Sy).*Fv)...
            +sum(sum(v'.*t2.*v,1))...
            +sum(sum(v'.*t3.*Sy,1))...
            -sum(sum(RSy'.*t4.*v,1)))/ej1...
            -(sum(sum(v'.*tempj.*v,1))-sum(K.*(v.^2)))*ej1j2/ej1^2-1;
        Brholam=(2*sum(v.*K.*HRWy)-sum((tempj*HRWy).*v)...
            -sum((tempj*v).*HRWy))/ej1...
            -(sum(sum(v'.*tempj.*v,1))-sum(K.*(v.^2)))*eji/ej1^2;
        t1=M'*HR;
        t1=R'*Hrho*R-(t1+t1');
        Blamrho=(sum(Wy.*(t1*y))...
            +2*sum(v.*D.*((H*M-Hrho*R)*Sy))...
            -sum(Dij.*(v.^2)))/di...
            -(sum(HRWy.*Ry)-sum(D.*(v.^2)))*dij/di^2;
        B=[[Blam,Blamrho];[Brholam,Brho]];
        g=2*(B')*[flam;frho]; % may want to devide by n
    end
end
```

```
% Cacluate variance matrix of II estimator of [beta;lam;rho]
function V=IIV(theta,y,X,W,M)
    [n,k]=size(X);
    beta=theta(1:k);
    lamrho=theta(k+1:k+2);
    Xb=X*beta;
    [¬,¬,B,D,F,G,H,K,R,GR,XRRXinvXR,v]=IIsolve(lamrho,y,X,W,M);
    Sigma=v.^2; % as a column vector
    RGXb=R*G*Xb;
    HRGXb=H*RGXb;
    HRGR=H*R*GR;
    Ed=tr(HRGR'.*Sigma,HRGR)+sum(HRGXb.^2);
    En2=HRGXb'*(HRGXb.*Sigma); % second part of numerator in Xi_lam
    E=HRGR-diag(D);
    SigmaE=E.*Sigma;
    En1=tr(SigmaE, (E+E').*Sigma);
    Xi_lam=(En1+En2)/(Ed^2);

    J1=XRRXinvXR*RGXb;
    J2=XRRXinvXR*(HRGXb.*Sigma)./Ed;
    L=F-diag(K);
    t1=tr(F'.*Sigma,F); % row vector of tr(Sigma*Fi'*Fi)
    SigmaL=L.*Sigma;
    t2=tr(SigmaL,(L+L').*Sigma);
    Xi_rho=t2/(t1^2);
    tt2=tr(SigmaE,(L+L').*Sigma);
    Xi_lamrho=tt2/(Ed*t1);
    Xi_rholam=Xi_lamrho;

    Xi=n.*[[Xi_lam,Xi_lamrho];[Xi_rholam,Xi_rho]];
    Binv=inv(B);
    V1=(B\Xi)/(B'); % variance of sqrt(n)*(lamrhoII-lamrho)
    V2=n.*XRRXinvXR*((XRRXinvXR').*Sigma);
    J1BJ2=J1*Binv(1,1)*(J2');
    V2=V2+J1*V1(1,1)*(J1')-n.*(J1BJ2+J1BJ2');
    Vlamb=n.*Binv(1,1)*(J2')-V1(1,1)*(J1');
    Vrhob=n.*Binv(2,1)*(J2')-V1(2,1)*(J1');
    V12=[Vlamb;Vrhob];
    V=[[V2,V12'];[V12,V1]]./n;
end

function [c,ceq]=fminconstrII(lamrho,wm)
    c1=abs(lamrho(1))-(wm(1)-0.0001);
    c2=abs(lamrho(2))-(wm(2)-0.0001);
    c=[c1;c2]; % parameter nonlinear inequality constraints
    ceq=[]; % set II as equality constraints
end

% trace function of product of two comformable matrices
function t=tr(A,B)
    t=sum(dg(A,B));
end

% diagonal elements of product of two comformable matrices
function d=dg(A,B)
    d=sum(A.*(B'),2);
end
```

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